Math 1A Fall 2014
Practice Final Name: ________________________________

No calculators. You have 3 hours.

Problem 1 For each of the following, either sketch an example of a function with the required properties, or tell me why none exist.

1. Surjective (onto) and injective (one-to-one)
   Every horizontal line should pass through our graph exactly one time. Examples: $f(x) = mx + b$, $f(x) = x^3$.
2. Odd and not surjective
   We need $f(-x) = -f(x)$ and that there is some value y such that $y = f(x)$ has no solution. $f(x) = \frac{1}{x}$ works.
3. Even and injective (one-to-one)
   We need $f(x) = f(-x)$ and that for every horizontal line ($y = c$), the equation $c = f(x)$ has at most one solution. This fails terribly. If $c = f(x)$, then $c = f(-x)$, so the function and the line intersect at least twice.
4. Not even, not odd, not surjective (onto), and not injective (one-to-one).
   We should take a function that is neither surjective nor injective ($f(x) = x^2$), and translate it so that it's also not symmetric. Try $f(x) = (x-1)^2$
5. Even and odd.
   We get $f(x) = f(-x) = -f(x)$. What numbers are their own negation? Just 0. The only function that works is $f(x) = 0$.

Problem 2 Show that $\lim\limits_{x \to 0} x^2 = 0$ by producing a $\delta - \epsilon$ proof.
Throughout we use that $f(x) = x^2$ means that $f(0) = 0$.

$$\text{If } |x^2 - 0| < \epsilon, \text{ then } |x - 0| < \sqrt{\epsilon}.$$

Let's try $\delta = \sqrt{\epsilon}$. Formally: Fix $\epsilon > 0$. Let $\delta = \sqrt{\epsilon}$.

$$|x - 0| < \delta \implies |x|^2 < (\sqrt{\epsilon})^2 \implies |x^2 - 0| < \epsilon.$$

Thus we can conclude that x^2 is continuous at zero.

Problem 3 Show that $f(x)$ is continuous on $(-\infty, \infty)$ (no $\delta - \epsilon$ work needed) for

$$f(x) = \begin{cases} \sin(x) & : x < \frac{\pi}{4} \\ \cos(x) & : x \geq \frac{\pi}{4} \end{cases}$$

Wee need to see that, at each point, the left hand limit is the same as the right hand limit, and both match the value of the function at that point. This is obvious for $x \neq \frac{\pi}{4}$ since sine and cosine are continuous. Let's check $x = \frac{\pi}{4}$.

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin(x) = \frac{\sqrt{2}}{2}$$

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} \cos(x) = \frac{\sqrt{2}}{2}$$

$$f(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

So f is continuous at $\frac{\pi}{2}$.

Problem 4 Prove the following: If f and g are two functions that satisfy $f' - g' = 0$ everywhere, and $f = g$ somewhere, then $f = g$ everywhere.

Let $h = f - g$. Then $h(x) = 0$ if and only if $f(x) = g(x)$. We also have that $h' = f' - g' = 0$ and $h(a) = f(a) - g(a) = 0$ for some value a. Our goal is now to show that $h = 0$ everywhere. We proceed by mean value theorem, and proof by contradiction.

Assume (for contradiction) that $h(x)$ is not the zero function. Then there is a point b such that $f(b) \neq 0$.

$$\text{Then } m = \frac{h(b) - h(a)}{b - a} = \frac{h(b)}{b - a} \neq 0.$$

By the Mean Value Theorem, there's a point c such that $h'(c) = m \neq 0$. But we know that $h'0$ for this problem, so this is a contradiction. Thus our assumption is flawed, and b such that $h(b) \neq 0$ cannot exist. Thus $h = 0$ everywhere.

Problem 5 The equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ defines a curve called an astroid. Find all the points on the curve where there is no tangent line.

This curve was harder than I wanted it to be, and shows a highly exceptional case. Carrying out implicit differentiation, you get two equations.

$$\frac{dy}{dz} = \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\frac{dx}{dy} = \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}.$$

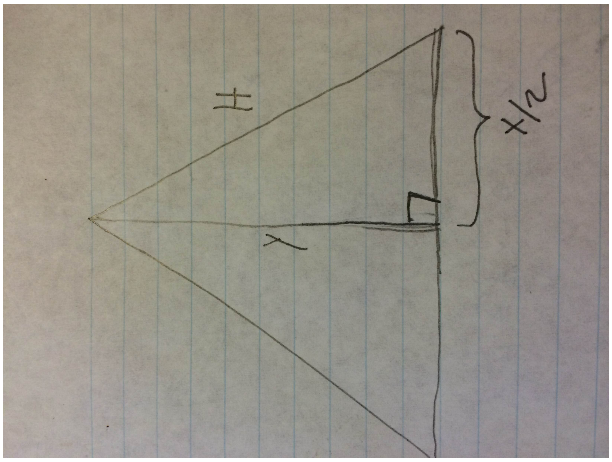
Obviously the points of interest are where the denominators are zero, so $(0, \pm 8)$ or $(\pm 8, 0)$. Based SOLELY on this information, a reasonable (yet largely incorrect) interpretation would be that there is a horizontal tangent at $(0, \pm 8)$ and a vertical tangent at $(\pm 8, 0)$.

To really see the solution, you would have to sketch the astroid. Follow this [link] to see a picture. It's up to interpretation whether there is, say, a vertical tangent at that top point. If we look at just the information to the left of the y-axis, then it makes sense. But when considered as a whole picture, we clearly cannot take a derivative at a sharp edge.

TL;DR Whether or not there are tangent lines at the four points is up to interpretation. Don't worry about anything this insane for your final.

Problem 6 Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is the equilateral triangle.

Optimizing with respect to angle turn out to be quite difficult. Let's do it by labeling as in this picture. Sorry it's sideways.



Then we have constant perimeter $P = x + 2H$ and nonconstant area $A = \frac{1}{2}xy$. Let's solve for y in terms of x and perimeter.

$$\begin{aligned} P =& 2H + x = 2\sqrt{(\frac{x}{2})^2 + y^2} + x \\ (P-x)^2 =& 4[(\frac{x}{2})^2 + y^2] \\ \frac{(P-x)^2}{4} - (\frac{x}{2})^2 =& y^2 \\ \frac{1}{4}[P^2 - 2Px] =& y^2 \end{aligned}$$

Now we plug this into A to get a function of one variable, and optimize.

$$\begin{aligned}
A &= \frac{1}{2}xy \\
A^2 &= (\frac{1}{2}x)^2y^2 \\
&= (\frac{1}{2}x)^2(\frac{1}{4}[P^2 - 2Px]]) \\
\frac{d}{dx}A^2 &= \frac{1}{16}[2P^2x - 6Px^2] \\
0 &= Px - 3x^2 \\
x &= \frac{P}{3}, 0
\end{aligned}$$

$x = 0$ does not correspond to a triangle. $x = \frac{P}{3}$ means that one-third of the perimiter goes into x. Since the triangle is isosceles, $H = \frac{P}{3}$ as well. So this is an equilateral triangle.

Problem 7 Compute the area under the curve $f(x) = x^3$ on $[0, 1]$ using Riemann sums. You will get no points for computing the integral using the usual shortcut.
Hint: Start by computing left and right -sided Reimann sums, and show that they approach the same value for large n. You will need this fact: $1^3 + 2^3 + \ldots + n^3 = [\frac{n(n+1)}{2}]^2$.
We compute the left and right side Riemann sums. We will find that they are the same, and will conclude that the sum is the area under the curve. this is valid because x^3 is increasing.

I'll write out the left side sum. the work for the right side sum is essentially the same.

$$\begin{aligned}
\sum_{i=0}^{n-1} f(\frac{i}{n})\frac{1}{n} &= \sum_{i=0}^{n-1}(\frac{i}{n})^3\frac{1}{n} \\
&= \frac{1}{n^4}\sum_{i=0}^{n-1} i^3 \\
&= \frac{1}{n^4}(0^3 + 1^3 + \ldots (n-1)^3) \\
&= \frac{1}{n^4}[\frac{(n-1)n}{2}]^2 \\
&\overset{n\to\infty}{\longrightarrow} \frac{1}{4}
\end{aligned}$$

So the sum approaches $\frac{1}{4}$.

Problem 8 Compute the derivative with respect to x of

$$y = \int_{\cos(x)}^{\sin(x)} ln(1 + 2v)\, dv$$

Fundamtental theorem of calculus. We get

$$\ln(1 + 2\sin(x))\cos(x) - \ln(1 + 2\cos(x))(-\sin(x))$$

Problem 9 Compute

$$\int_e^{e^4} \frac{dx}{x\sqrt{ln(x)}}$$

Try $u = \ln(x)$, $du = \frac{dx}{x}$. Then we get

$$\int_1^4 u^{-\frac{1}{2}} du$$

$$2u^{\frac{1}{2}}]_1^4 = 2$$

Problem 10 The curves $y = x^2$ and $y = 2 - x^2$ bound a region. Use the method of cylindrical shells to rotate this region about the line $x = 1$ and find the volume.
The integral is

$$2\pi \int_{-1}^{1} (1 - x)[(2 - x^2) - x^2] dx = \frac{16\pi}{3}$$