Summation Problem

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The problem states that you should compute

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i-2}{n}\right)^2+1\right]\left(\frac{2}{n}\right)$$

There are two ways of doing this. One is to take advantage of several summation formulas that you may know, the other is to identify this as a Riemann sum, find the corresponding integral, and compute.

Method 1

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i-2}{n}\right)^2+1\right]\left(\frac{2}{n}\right) &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i^2-8i+4}{n^2}\right)+1\right]\left(\frac{2}{n}\right)\\ &= \lim_{n\to\infty}\left[\left(\frac{4\frac{n(n+1)(2n+1)}{6}-8\frac{n(n+1)}{2}+4}{n^2}\right)+n\right]\left(\frac{2}{n}\right)\\ &= \frac{14}{3}\end{aligned}$$

For this method, you needed to know the sums of the first n numbers, and the first n squares. Obviously this means being very comfortable with sigma notation. You also needed to evaluate a fairly large (but straightforward) limit at the end.

Method 2

We want to make the identification

$$\int_a^b f(x)\,dx \sim \sum f(x_i)\Delta x = \sum_{i=1}^{n}\left[\left(\frac{2i-2}{n}\right)^2+1\right]\left(\frac{2}{n}\right)$$

for some function f with boundary a and b.

We gather information about what the sum will look like as an integral. We see on the right that $\Delta x = \frac{b-a}{n} = \frac{2}{n}$. Let's say $a = 0$ and $b = 2$. It's generally easiest to pick $a = 0$. This sum has this annoying $\frac{2}{n}$ term on the inside that's hard to work with. Most of these problems (at least the ones in the book) don't. We can get rid of it by changing our index i.

$$\sum_{i=1}^{n}\left[\left(\frac{2i-2}{n}\right)^2+1\right]\left(\frac{2}{n}\right)=\left[\left(\frac{0}{n}\right)^2+1\right]\left(\frac{2}{n}\right)+\left[\left(\frac{2}{n}\right)^2+1\right]\left(\frac{2}{n}\right)+\left[\left(\frac{4}{n}\right)^2+1\right]\left(\frac{2}{n}\right)\dots$$
$$=\sum_{i=0}^{n-1}\left[\left(\frac{2i}{n}\right)^2+1\right]\left(\frac{2}{n}\right)$$

That was the hard part. Now the sum is much easier to work with. Since $\Delta x = \frac{2}{n}$, we identify $x*$ as $\frac{2i}{n}$.

Now it's easy to pick out f. What is $f(\frac{2i}{n})$ in the sum? $f(\frac{2i}{n}) = \left[\left(\frac{2i}{n}\right)^2 + 1\right]$. Then we have $f(x) = \left[(x)^2 + 1\right]$, and $(a, b) = (0, 2)$. So we just compute

$$\int_0^2 x^2 + 1\,dx = \frac{1}{3}(2^3 - 0^3) + (2 - 0) = \frac{14}{3}$$

In this case, the first method looks easier. However, the second method is a powerful tool for computing integrals, and it more closely relates to the material from this semester.