Homework 1: Rotations and Transformations

Due: Fri, 9/12

Set: August 29, 2014

Problem-sets are due in before 11:59 pm on the day they are due via BSpace (or via email if you are unable to use BSpace) Feel free to use a computer to help you with this problem set. If you do write any code, to help you solve the problem, attach the code at the end of your problem set. If you use any pre-made code (such as MATLAB's pseudo-inverse function pinv()), state that you use it as a step in your solution.

Problems:

This problem set focuses on the different methods for representing rotations. These methods are important for understanding the orientation of objects. In this problem set we investigate two common devices: servos (rotational actuators with angle encoders) and gimbals (seen in ships, spacecraft and in fancy cup-holders).

1 Building a Rotation from a Basis

We begin by showing how you can construct a rotation matrix from a set of bases both in 2D and 3D.

1. Consider the two pairs of vectors:

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$$

   (a) Suppose $\boldsymbol{x}_V = \begin{bmatrix} x_V & y_V \end{bmatrix}^T$ are the corodinates of some vector $\boldsymbol{v}$ written with respect to the $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ frame. Find a transformation matrix T, such that $\boldsymbol{x}_U = T\boldsymbol{x}_V$, where $\boldsymbol{x}_U$ are the coordinates of the same vector in the $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ frame.

   (b) Is T a valid rotation matrix? If so what is the angle of rotation, if not why not.

2. Consider the two pairs of vectors:

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \right\} \qquad \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

   (a) Given the coordinate $\boldsymbol{x}_V$ written with respect to the $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ frame, write a transformation T, such that $\boldsymbol{x}_U = T\boldsymbol{x}_V$, where $\boldsymbol{x}_U$ are the coordinates of the same vector in the $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ frame.

   (b) Is T a valid rotation matrix? If so what is the angle of rotation, if not why not.

You can check your results by plotting coordinates in each of these frames as shown in the Appendix 2 in the reader.

2 Servos and Rodrigues' formula

A servomotor or *servo* is a rotary actuator combined with a rotary encoder which provides position control. They are heavily used in robotics and automation and can range from micro hobby servos weighing under five grams to several tonne devices.
We wish to find the position of a point attached to the top of a servo motor. We know the position of this point in the servo output frame and denote it $\boldsymbol{p}_S$. We also know the axis about which the servo rotates in the world frame, which we denote $\boldsymbol{\omega}$, as well as the angle by which it has rotated θ. Given this information, we wish to find the position of this point in the world frame $\boldsymbol{p}_W$

1. For an arbitrary ω and θ, write an expression for the resulting rotation matrix R.
2. Consider a servo with the rotational axis pointing along the z-axis $\left(\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T\right)$ (Figure 1 left). If it rotates by $\theta = 45$ degrees, what is the corresponding rotation matrix. Given a point with coordinates $\boldsymbol{p}_S = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$, what is the corresponding position in the world frame?
3. Dr Kerman sees you experiment and decides to kick your servo. The servo axis misaligns and now points along the vector given by $\left(\begin{bmatrix}0.5 & 0.2 & 2\end{bmatrix}^T\right)$ (Figure 1 right). You remain unphased, and start computing a new rotation matrix. If the servo rotates by $\theta = 25$ degrees, what is the corresponding rotation matrix.

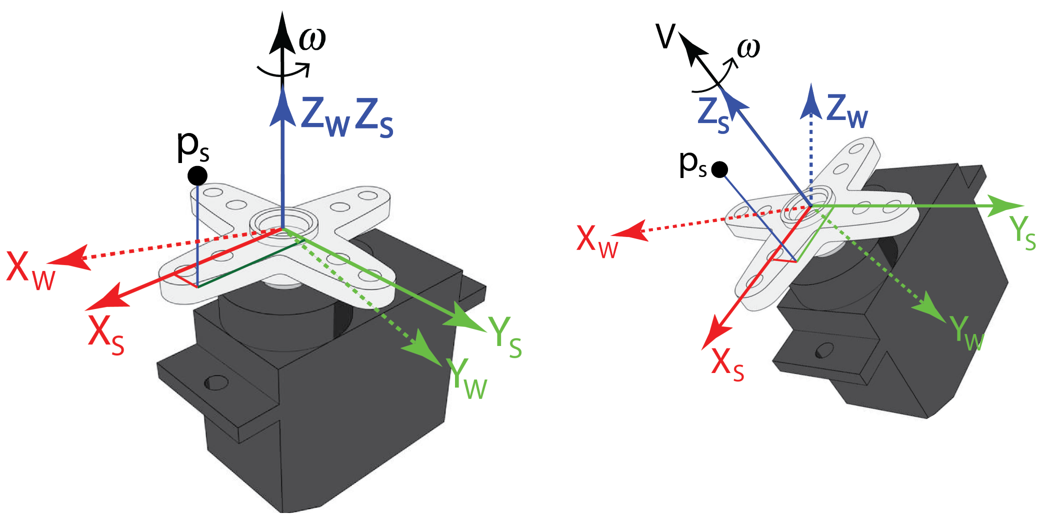

Figure 1: Illustration of the servo with world $\{\boldsymbol{x}_W, \boldsymbol{y}_W, \boldsymbol{z}_W\}$ and servo $\{\boldsymbol{x}_S, \boldsymbol{y}_S, \boldsymbol{z}_S\}$ coordinate frames shown. The servo's axis of rotation is shown as ω and a reference point is shown.

3 Gimbals and Too Many Rotations

A *gimbal* (actually a triple gimbal) is a set of three disks that are able to rotate independently from one another. In doing so, the rotation of the innermost ring is completely independent from the rotation of the base. They are often used for rotational isolation and for attitude measurement in boats, aircraft and spacecraft.
You decide to visit Dr Kerman to see what project they have been working on that has resulted in them kicking over your experiment. They have been trying to turn the Campanile into a rocket, and have been using a gimbal to measure the relative rotation between the Campanile-rocket and the ground. The gimbal returns the rotation matrix R_{WC}, the rotation that maps coordinates from the Campanile frame to the world frame. Dr Kerman has been converting the R_{WC} rotation matrix into *ZYX Euler Angles* also known as *yaw-pitch-roll*. However the resulting angles have strange discontinuities and they are unsure as to why. You decide to help Dr Kerman to prevent any more lab damage.

1. Your first debugging step is to check Dr Kerman's angle recovery program. Dr Kermans algorithm converts the rotation matrix $R_{WC} \in \mathbb{R}^{3\times3}$ into the three Euler angles $\{\theta_X, \theta_Y, \theta_Z\} = \left\{-\frac{\pi}{8}, -\frac{\pi}{2}, \frac{11\pi}{24}\right\}$ using the Z-Y-X convention.
   (a) Given these Euler angles, reconstruct the corresponding R_{WC} matrix.
   (b) Compare this rotation matrix with the one generated by the Euler angles: $\left\{-\frac{\pi}{3}, -\frac{\pi}{2}, \frac{2\pi}{3}\right\}$ and $\left\{\frac{\pi}{12}, -\frac{\pi}{2}, \frac{\pi}{4}\right\}$. Can you explain what is going on?
2. You decide to use quaternions to represent the matrix R_{WC}.
   (a) Write the rotation matrix you found using Euler angles: $\{\theta_X, \theta_Y, \theta_Z\} = \left\{-\frac{\pi}{8}, -\frac{\pi}{2}, \frac{11\pi}{24}\right\}$ in quaternions.
   (b) Consider the other two rotation matrices you found in the first part (i.e. for $\left\{-\frac{\pi}{3}, -\frac{\pi}{2}, \frac{2\pi}{3}\right\}$ and $\left\{\frac{\pi}{12}, -\frac{\pi}{2}, \frac{\pi}{4}\right\}$). Explain any similarities or differences in the Euler Angles, Rotation Matrix and Quaternion rotation conventions.

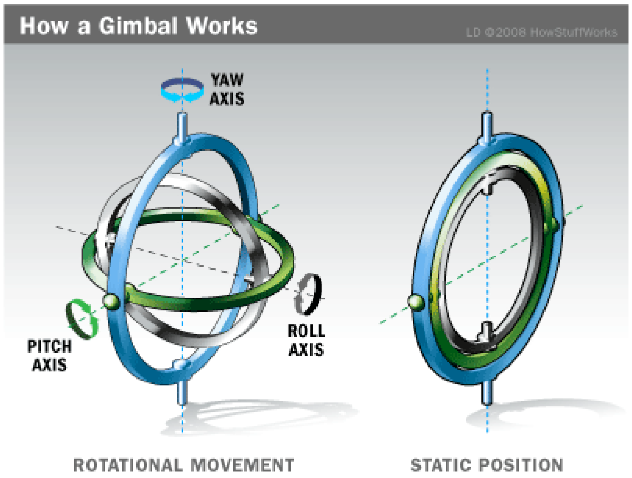

Figure 2: Illustration of a gimbal

Bonus Dr Kerman sees asks you to tell the time currently shown on the Campanile's clock-face. He makes the problem more difficult by only giving you three rotation matrices- R_{WC}, R_{WH}, R_{WM} the rotation matrices that convert from the Campanile frame, the Hour hand frame and the Minute hand frame. Using the below rotation matrices, determine:

- The time (will be ambiguous to am/pm).
- If you will go into space today (which end is pointing up?).

$$R_{WC} = \begin{bmatrix} 0.5590 & 0.8255 & 0.0773 \\ 0.7694 & -0.4818 & -0.4194 \\ -0.3090 & 0.2939 & -0.9045 \end{bmatrix}$$

$$R_{WH} = \begin{bmatrix} 0.5590 & -0.2883 & 0.7774 \\ 0.7694 & 0.5298 & -0.3568 \\ -0.3090 & 0.7976 & 0.5180 \end{bmatrix}$$

$$R_{WM} = \begin{bmatrix} 0.5590 & 0.7228 & 0.4064 \\ 0.7694 & -0.2695 & -0.5791 \\ -0.3090 & 0.6364 & -0.7068 \end{bmatrix}$$

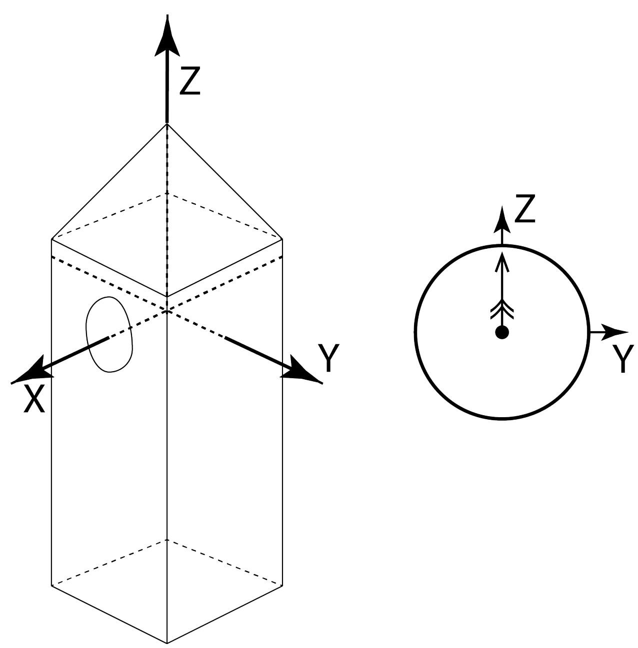

Figure 3: Coordinate frame of the Campanile and clock-face