Robert Peter Matthew

Introduction to Robotics

EE C125/EE 215A/BIOE C125

Schedule of Classes

Class		Lecture	Discussion	Homework
Number	Date			
L0	Aug 28	Organisation	—	
L1	Sept 02	Rigid Body Motion	L0,1	HW1
L2	Sept 04	Rotation Matrices	—	L1-3
L3	Sept 09	Homogenous Coordinates	L2,3	
L4	Sept 11	Exponential Coordinates	—	—
L5	Sept 16	Forward Kinematics	L4,5	HW2
L6	Sept 18	2D Vision	—	L3-5
L7	Sept 23	3D Vision	Review	
L8	Sept 25	Rigid Body Velocities	—	—
L9	Sept 30	Inverse Kinematics	HW1,2	
L10	Oct 02	Review	—	—
L11	Oct 07	MIDTERM 1 L1-5	L6,7	HW3
L12	Oct 09	Jacobians	—	L8-12
L13	Oct 14	Path Planning Intro	L8,15	
L14	Oct 16	Path Planning Algorithms	—	—
L15	Oct 21	Force Wrenches	L16,17	HW4
L16	Oct 23	Inertial Properties	—	L15-18
L17	Oct 28	Newtonian Dynamics	L18	
L18	Oct 30	Lagrangian Dynamics	—	
L19	Nov 04	Dynamics of Open Chains	Review	
L20	Nov 06	Dynamics of Open Chains	—	
L21	Nov 11	Holiday	HW3,4	—
L22	Nov 13	Review	—	
L23	Nov 18	MIDTERM 2 L8-12,15-18	Projects	
L24	Nov 20	Feedback Control	—	
L25	Nov 25	Medical Robotics	Projects	
L26	Nov 27	Holiday	—	
L27	Dec 02	Projects	Projects	
L28	Dec 04	PROJECT REVIEWS	—	

Course Information

Instructors

Role	Name	Email	Office Hours	Location
Professor	Ruzena Bajcsy	bajcsy@eecs	MW 0900-1000	719 SDH
GSI	Aaron Bestick	abestick@eecs	-	-
GSI	Austin Buchan	abuchan@eecs	-	-
GSI	Robert Matthew	rpmatthew@eecs	M 10-11, 15-16	337A Cory

Lectures

Day	Time	Instructor	Location
Tu Th	1400-1530	Ruzena Bajcsy	521 Cory

Labs

Day	Time	Instructor	Location
W	1100-1400	Austin Buchan	119 Cory
W	1400-1700	Austin Buchan	119 Cory
Th	1100-1400	Aaron Bestick	119 Cory
F	1200-1500	Aaron Bestick	119 Cory

Discussions

Day	Time	Instructor	Location
Tu	1000-1100	Robert Matthew	293 Cory
W	1000-1100	Robert Matthew	247 Cory

Grading:

Labs 20%

There will be a total of eight labs spaced over the start of the semester. Labs will ask you to complete a number of exercises, including derivation of equations as implementation in Python, ROS, OpenCV for use with cameras, the Baxter robot and other hardware. Performance in the labs will be based on both attendance and a lab report.

Homeworks 20%

The timetable has been structured so that you have at least one dedicated discussion section before the deadline, and to ensure that you get a graded assignment in time for the midterms.

No extensions will be allowed.
Please plan accordingly.

Homework	Set Date	Due Date	Return Date
1	Aug 29	Sept 12	Sept 18
2	Sept 12	Sept 26	Oct 02
3	Oct 10	Oct 24	Oct 30
4	Oct 24	Nov 7	Nov 13

The homework grade has two components:

Solutions: 40 points Homework solutions should be clear and complete, demonstrate important intermediate steps, and provide references to the textbook or outside materials as appropriate. Students may work together on the assignments, but each individual must write their own solutions independently.

Presentation: 10 points The homework writeup must be legibly hand written or typeset and free of excessive erasing, scratched-out text, etc. We urge students to solve the homework on scratch paper, then copy completed solutions onto new sheets of paper.

Questions about grading should be sent to the GSIs.

Exams 30%

Before each exam, you should have your relevant homeworks returned, and both an inclass review and dedicated discussion section.
Please let us know in advance if you are unable to make any of the dates below or require any special arrangements.

	Date	Topics	Location
Midterm 1	Oct 07	Lecture 1-5	Inclass
Midterm 2	Nov 18	Lecture 8-12,15-18	Inclass
Final		None	

Final Project 30%

A substantial part of your grade will depend on the final group project. Groups comprise of 2 or 3 people, and can include people outside of your lab group.
The choice of project topic is up to you, but the final project must integrate perception, planning, and action in some non-trivial way.
There are three milestone deliverables for the projects. The first is a two side project brief that includes the team, the task and an outline of your project. Details on required parts and materials and initial analysis of the problem is required.
The bulk of the project is allocated later in the semester. Projects will be graded based on a presentation & Demo during dead week, and a final report. The final report will be in the form of a website that will go live on the final day of the semester.

Deliverable	Deadline	Weighting
Project Brief	Oct 16	5 %
Presentation & Demo	Dec 12	10 %
Final Report (Website)	Dec 19	15 %

Students with special requirements

If you have been issued a letter of accommodation from the Disabled Students Program (DSP), please contact either Professor Bajcsy or a GSI as soon as possible to work out the necessary arrangements. If you need an accommodation and have not yet seen a Disability Specialist at the DSP, please do so as soon as possible.
If you would need any assistance in the event of an emergency evacuation of the building, the DSP recommends that you make a plan for this in advance. (Contact the DSP access specialist at 643-6456.)

Discussions

Discussion sections are designed to provide supplement the lectures by highlighting key ideas and providing sample questions. Attendance is not mandatory, though it is recommended. Class participation is encouraged and questions welcome.

Labs

Laboratory sections are used to provide hands on experience with some the techniques presented in the course. Attendance is mandatory and is limited to students registered for that particular lab section. This restriction is due to limited resources in the labs. The lab grade has three components:

Participation: 10 points Students are expected to exhibit friendly, active involvement in the lab, and to keep the laboratory facilities clean and functional throughout the semester.

Tasks, questions, and code: 30 points Complete each task specified in the lab assignment, answer lab questions thoughtfully, and provide supporting mathematics and references where appropriate.

Report: 10 points The laboratory writeup should be a self-contained document providing detail about each task and question. It should be written in complete sentences with full questions statements; however, information about the lab setup and motivation is unnecessary. All figures need clear captions, legends, and labels, and must be readable when printed in grayscale. Matlab code should be included in an appendix.

Questions about grading should be sent to the GSIs.

Conduct

This course aspires to be a safe space for all students regardless of age, sex, gender, race, nationality, ability etc. Any discriminatory behavior, or failure to respect personal boundaries will result in removal from the course. If you feel harassed my any member of the instructional team, a colleague or anyone, bring it to the attention of a trusted instructor, counselor, or campus police.

UCB Police police.berkeley.edu

Sexual Harassment & Survivor Support survivorsupport.berkeley.edu

University Health Services- Social Services uhs.berkeley.edu

Gender Equity Resource Center geneq.berkeley.edu

Office for the Prevention of Discrimination and Harassment ophd.berkeley.edu

Introduction

These notes are designed to act as a companion to the Introduction to Robotics course and the course text *A Mathematical Introduction to Robotic Manipulation*. The notes are still under development and will be updated regularly. There will be errors, flaws and issues with the way the course materials are presented. Any comments, changes, requests and improvements are greatly welcome.
These notes are based on the invaluable notes on Linear Systems (221A) by Claire Tomlin and the hand written tome of Robotics notes by Dan Calderone. Without these people, these notes would not exist.
Where possible I have used examples taken from a number of seminal mathematical, robotics and dynamics textbooks. They are listed in the bibliography, and referenced throughout.

Chapter 1

Rotations (Chapter 2.2 in MLS)

1.1 General Rotations

- Rotations describe a transformation between two frames.
- We use the notation R_{AB} to describe the rotation of frame B in frame A.
- Frames and rotations follow the right hand rules.
- Rotations are a mathematical group. Therefore they have a number of nice properties:
  - They are associative: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
  - There is an identity element $\mathbb{I}$ which satisfies $R\mathbb{I} = R = \mathbb{I}R$
  - For every rotation R there is a unique inverse rotation R^{-1} where: $RR^{-1} = R{-}1R = \mathbb{I}$
- Rotations can be be combined in sequence to form new rotation matrices. Given the rotations of frame B in the A frame (R_{AB}) and the rotation for C in the B frame (R_{BC}), we can write and expression for the rotation of frame C in A: $R_{AB}R_{BC} = R_{AC}$.

1.2 Rotation Matrices

- Rotation matrices have two key properties, they are *orthonormal* and *orthogonal* with a determinant of 1. (Do not worry if this is new to you, there is a quick review in Appendix A.1).
- Matrices that satisfy these properties are called Special Orthogonal Group (SO).
- Rotations in 2D are characterised by a 2×2 matrix. This is written $R_{AB} \in \mathbb{R}^{2\times 2}$. It is also a member of the Special Orthogonal Matrices 2 ie. $R_{AB} \in SO(2)$.
- Rotations in 3D are characterised by a 3×3 matrix. This is written $R_{AB} \in \mathbb{R}^{3\times 3}$. It is also a member of the Special Orthogonal Matrices 3 ie. $R_{AB} \in SO(3)$.

- The identity identity element $\mathbb{I}$ are the $\mathbb{I}_2$ and $\mathbb{I}_3$ identity matrices.
- As the columns of R are mutually orthonormal, $RR^T = R^T R = \mathbb{I}$.
- This means that the inverse R^{-1} for an rotation matrix R is its transpose R^T.

1.3 Euler Angles

- Probably the most common way to represent rotations, Euler angles are used to represent a rotation θ about the basis vectors of a frame ($\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$).
- Each Euler rotation is defined by the axis of the rotation and the angle:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix}$$

$$R_z(\alpha) = \begin{bmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- When specifying a combination of Euler Angles, the order of rotation MUST be given. Rotations *are not commutative*: $R_1 R_2 R_3 \neq R_1 R_3 R_2$.
- Euler angles are not a perfect representation of a rotation matrix. Ambiguities can arise when axes align. These are called singularities and can present problems. Singularities arise for ANY three dimensional representation of SO(3).

1.4 Exponential Coordinates

- Exponential Coordinates represent a rotation as an axis of rotation $\boldsymbol{\omega}$ and an angle of rotation θ:

$$R\left(\boldsymbol{\omega}, \theta\right) = e^{\hat{\boldsymbol{\omega}}\theta}$$

  where the $\hat{}$ is the 'hat' operator, a matrix form of the cross product.

- Being a 4 dimensional descriptor of a rotation, it does not experience issues with singularities.

- We can show how this representation is created via the construction of the Rodrigues formula.

Hat Operator

- We use the hat operator as a convenient matrix way of writing a cross product of two vectors:

$$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{a}}\boldsymbol{b}$$

- We define $\hat{\boldsymbol{a}}$ as:

$$\hat{\boldsymbol{a}} = \widehat{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Note that $\hat{\boldsymbol{a}}$ is *skew symmetric* i.e. $\hat{\boldsymbol{a}}^T = -\hat{\boldsymbol{a}}$

Rodrigues' Formula

The derivation of the Rodrigues' formula is given in chapter 2.2.2 of MLS. We include the formula here for completeness:

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{w}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} sin\left(\|\omega\|\,\theta\right) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}\left(1 - cos\left(\|\omega\|\,\theta\right)\right)$$

We can also find the exponential representation of any rotation matrix given the expression:

$$\theta = cos^{-1}\left(\tfrac{trace(R)-1}{2}\right)$$

$$\boldsymbol{\omega} = \tfrac{1}{2sin(\theta)}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

note that if $\theta = 0$ then the definition of $\boldsymbol{\omega}$ is arbitrary.

1.5 Quaternions

- Quaternions are another 4 dimensional representation of rotations.
- A quaternion $\boldsymbol{Q}$ is an extended imaginary vector of the form:

  $$\boldsymbol{Q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$

  with $q_i \in \mathbb{R}$
- Multiplication is associative and distributive, but NOT commutative.
- The conjugate of a quaternion $\boldsymbol{Q} = (q_0, \boldsymbol{q})$ is written $\boldsymbol{Q}^\star = (q_0, -\boldsymbol{q})$
- The magnitude of a quaternion $\boldsymbol{Q}$ is $\|\boldsymbol{Q}\|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$
- The inverse of a quaternion $\boldsymbol{Q}$ is written as $\boldsymbol{Q}^{-1} = \frac{\boldsymbol{Q}^\star}{\|\boldsymbol{Q}\|^2}$
- The identity quaternion is $(1, \mathbf{0})$
- Quaternions can be converted into rotations via $R = e^{\hat{\boldsymbol{\omega}}\theta}$ where:

  $$\theta = 2cos^{-1}q_0 \qquad \omega = \begin{cases} \frac{(q_1, q_2, q_3)}{sin\left(\frac{\theta}{2}\right)} & if\ \theta \neq 0 \\ 0 & otherwise \end{cases}$$

Chapter 2

Rigid Body Motion and Homogeneous Coordinates

In this chapter we outline the basics of Forward kinematics. We introduce general transformations between two frames, Homogeneous coordinate notation and how they can be used to compute the *Forward Kinematics* of a system.

2.1 General Transformations

In the previous discussion, we looked a pure rotations. These rotations acted as transforms about an common origin point.

$$\boldsymbol{q}_A = R_{AB}\boldsymbol{q}_B$$

Let us consider a *Rigid Body Transformation* between any two frames. This rigid body motion can be described as two components: a Rotation R_{AB} and a translation p_{AB}. Now any point in coordinate frame B, $\boldsymbol{q}_B$ can be written in coordinate frame A as $\boldsymbol{q}_A$ where:

$$\boldsymbol{q}_A = \boldsymbol{p}_{AB} + R_{AB}\boldsymbol{q}_B.$$

where $\boldsymbol{p}_{AB}$ is the translation from the origin in the A coordinate frame to the origin in the B coordinate frame.
The pair of both a translational component $\boldsymbol{p}_{AB} \in \mathbb{R}^3$ and a rotation $R_{AB} \in \mathcal{SO}(3)$ is known as a *configuration*. The *configuration space* is therefore the space of all translations and rotations combined. More formally, it is the product space between $\mathbb{R}^3$ and $\mathcal{SO}^3$. This is the *Special Euclidean Group*(3) denoted $\mathcal{SE}(3)$.
We can therefore write the action of any rigid transformation g_{AB} from the B coordinate frame to the A coordinate frame as:

$$\begin{aligned}\boldsymbol{q}_A &= g_{AB}\left(\boldsymbol{p}_{AB}, R_{AB}, \boldsymbol{q}_B\right)\\ \boldsymbol{q}_A &= \boldsymbol{p}_{AB} + R_{AB}\boldsymbol{q}_B \qquad (2.1)\end{aligned}$$

This is quite a cumbersome expression. However, we can simplify it substantially through the use of *Homogeneous Coordinates*.

2.2 Homogeneous Coordinates

Homogeneous coordinates allow for operations in $\mathcal{SE}(3)$ to be written as matrix operations in $\mathbb{R}^4$. This allows the combination of a rotation p_{AB} and R_{AB} to be written as a single 4×4 matrix, simplifying calculations. To do this we choose the following representations for points and vectors in $\mathbb{R}^3$.

Points Consider a point $\boldsymbol{q}_B \in \mathbb{R}^3$. We append a 1 to the end of this column vector to give the homogeneous coordinate representation:

$$\overline{\boldsymbol{q}}_B = \begin{bmatrix} q_{B,1} \\ q_{B,2} \\ q_{B,3} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors Vectors are the differences between two points. They will therefore have the representation:

$$\overline{\boldsymbol{v}} = \overline{\boldsymbol{q}}_B - \overline{\boldsymbol{q}}_A = \begin{bmatrix} \boldsymbol{q}_B - \boldsymbol{q}_A \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

where we note the 0 in the fourth row.

This notation is consistent with our definitions of points and vectors:

- The sum (and difference) of two vectors is another vector.
- The sum of a point and a vector is a point.
- The difference between two points is a vector.
- The sum of two points has no meaning.

We return out attention to Equation 2.1. By adopting this homogeneous coordinate notation, we find we can represent this rigid body transformation as:

Equation 2.1: $\boldsymbol{q}_A = \boldsymbol{p}_{AB} + R_{AB}\boldsymbol{q}_B$ as shown on Page 7

$$\overline{\boldsymbol{q}}_A = \begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} = \overline{\boldsymbol{g}}_{AB}\overline{\boldsymbol{q}}_B$$

We call $\overline{\boldsymbol{g}}_{AB}$ the homogeneous representation of g_{AB}.

2.3 Forward Kinematics

The *Kinematics* of a robotic manipulator describes the relationship between the motion of the joints and the motion of the rigid bodies that make up the manipulator. *Forward* Kinematics allows the configuration (position and orientation) of the end effector to be determined given a description of the joints and the joint angles.
We will restrict our attention (for now) to purely revolute joints, joints which have no translational component.

Example 2.3.1. Consider a planar two jointed manipulator show in in Figure 2.1. We consider the origin to be the centre of the world frame, and that joints one and two each have their own coordinate frame. The positions of the manipulators joints can be parametrised by $\boldsymbol{p}_{W1}$ and $\boldsymbol{p}_{12}$, the positions of joint 1 in the world coordinate frame, and the position of joint 2 in the first joints coordinate frame. The rotations about each joint are given by R_{W1} and R_{12}, the rotation of the first frame in the world frame, and the rotation of the second frame in the first frame respectively.

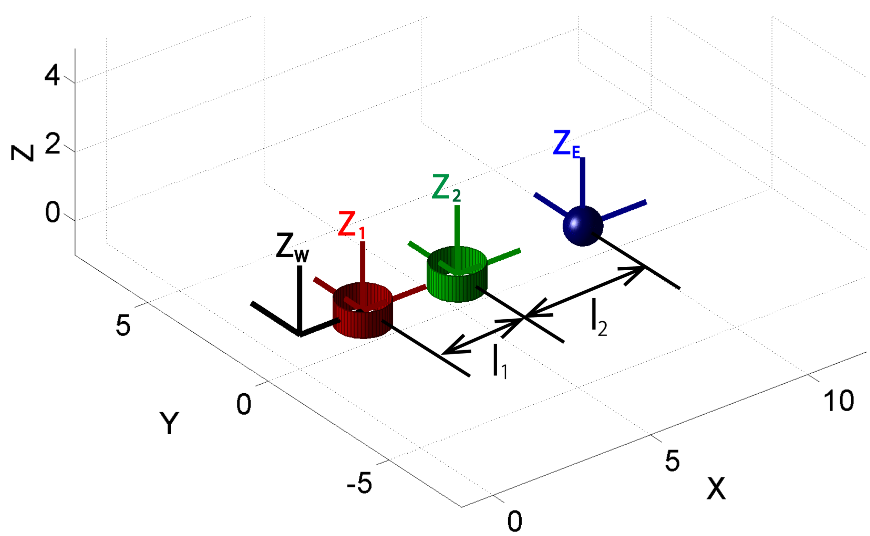

Figure 2.1: Cartoon of the two link planar manipulator, with world frame in black, joints one and two in red and green, and the end-effector in blue.

Consider the position of the end-effector given in frame two $\boldsymbol{q}_{2,E}$. The coordinates of this point in the first frame can be written using Equation 2.1 giving:

$$\boldsymbol{q}_{1,E} = \boldsymbol{p}_{1,2} + R_{1,2}\boldsymbol{q}_{2,E}$$

Equation 2.1: $\boldsymbol{q}_A = \boldsymbol{p}_{AB} + R_{AB}\boldsymbol{q}_B$ as shown on Page 7

Similarly this can be represented in the world coordinate frame as:

$$\boldsymbol{q}_{W,E} = \boldsymbol{p}_{W,1} + R_{W,1}\boldsymbol{q}_{1,E}$$

$$\begin{aligned}\boldsymbol{q}_{W,E} &= \boldsymbol{p}_{W,1} + R_{W,1}\left(\boldsymbol{p}_{1,2} + R_{1,2}\boldsymbol{q}_{2,E}\right) \\ \boldsymbol{q}_{W,E} &= \boldsymbol{p}_{W,1} + R_{W,1}\boldsymbol{p}_{1,2} + R_{W,1}R_{1,2}\boldsymbol{q}_{2,E} \end{aligned} \tag{2.2}$$

This expression can be written in a more compact form using homogeneous coordinates:

$$\overline{\boldsymbol{q}}_{W,E} = \overline{\boldsymbol{g}}_{W,1}\overline{\boldsymbol{g}}_{1,2}\overline{\boldsymbol{q}}_{2,E}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} R_{W,1} & \boldsymbol{p}_{W,1} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} R_{1,2} & \boldsymbol{p}_{1,2} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{2,E} \\ 1 \end{bmatrix}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} R_{W,1}R_{1,2} & R_{W,1}\boldsymbol{p}_{1,2} + \boldsymbol{p}_{W,1} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{2,E} \\ 1 \end{bmatrix}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} R_{W,1}R_{1,2}\boldsymbol{q}_{2,E} + R_{W,1}\boldsymbol{p}_{1,2} + \boldsymbol{p}_{W,1} \\ 1 \end{bmatrix}$$

which is the same as the expression shown in Equation 2.2.

Let us parametrise the manipulator as shown in the diagram. We know that both $R_{W,1}$ and $R_{1,2}$ are rotations about the z-axis and therefore have the form:

$$R_{W,1} = \begin{bmatrix} cos\left(\theta_1\right) & -sin\left(\theta_1\right) & 0 \\ sin\left(\theta_1\right) & cos\left(\theta_1\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{1,2} = \begin{bmatrix} cos\left(\theta_2\right) & -sin\left(\theta_2\right) & 0 \\ sin\left(\theta_2\right) & cos\left(\theta_2\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

similarly, the translation components are written:

$$\boldsymbol{p}_{W,1} = \begin{bmatrix} x_{W,1} \\ y_{W,1} \\ 0 \end{bmatrix} \quad \boldsymbol{p}_{1,2} = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{q}_{2,E} = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$$

This allows us to write the configuration of the end-effector in the word frame via:

$$\overline{\boldsymbol{q}}_{W,E} = \overline{\boldsymbol{g}}_{W,1}\overline{\boldsymbol{g}}_{1,2}\overline{\boldsymbol{q}}_{2,E}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} cos\left(\theta_1\right) & -sin\left(\theta_1\right) & 0 & x_{W,1} \\ sin\left(\theta_1\right) & cos\left(\theta_1\right) & 0 & y_{W,1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\left(\theta_2\right) & -sin\left(\theta_2\right) & 0 & l_1 \\ sin\left(\theta_2\right) & cos\left(\theta_2\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} cos\left(\theta_1\right) & -sin\left(\theta_1\right) & 0 & x_{W,1} \\ sin\left(\theta_1\right) & cos\left(\theta_1\right) & 0 & y_{W,1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 cos\left(\theta_2\right) + l_1 \\ l_2 sin\left(\theta_2\right) \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{\boldsymbol{q}}_{W,E} = \begin{bmatrix} l_2 cos\left(\theta_1 + \theta_2\right) + l_1 cos\left(\theta_1\right) + x_{W,1} \\ l_2 sin\left(\theta_1 - \theta_2\right) + l_1 sin\left(\theta_1\right) + y_{W,1} \\ 0 \\ 1 \end{bmatrix}$$

Chapter 3

Screw theory, Exponential Coordinates and Twists

In the past weeks, we have been building our intuition of rigid body motion. We now extend this intuition by introducing *Screw Theory* and its application to rigid body motion and robotics. By doing this we will develop a highly versatile formulation for Forward Kinematics that can describe joints that both rotate and translate simultaneously.

3.1 Screw Theory

Screw theory, and our motivations for using it can be summarised by *Chasles Theorem*:

Every rigid body motion can be realised by a rotation about an axis combined with a translation parallel to that axis.

This concept of a rotation and translation about a common axis can be visualised by any rotational helix. A common example of this is the thread on a screw (hence the name *Screw* theory). In Engineering, a screw thread is determined by a *pitch*, the ratio of translational motion to rotational motion. To describe the direction that the screw points, we need an *axis* that it points along. Finally to describe a point on the screw we need a *magnitude*, a measure of distance along the screw that will allow us to find our location on the helix. Therefore to describe a rigid body motion using a screw, we need three components: a *pitch*, *axis* and *magnitude*.
Keeping this 'screw' representation in mind, we will build up the mathematical formalism that can be used to create a 'screw' representation for rigid body motion.

3.2 Exponential Coordinates of Rotation

We start by looking at pure rotation about an axis $\boldsymbol{\omega}$. From mechanics, we can say that the velocity of a point $\boldsymbol{q}$ that is rotated by a *constant unit velocity* $\boldsymbol{\omega}$ is given by:

$$\dot{\boldsymbol{q}} = \boldsymbol{\omega} \times \boldsymbol{q}\left(t\right) \tag{3.1}$$

This is a differential equation which as the solution:

$$\boldsymbol{q}\left(t\right) = e^{\hat{\boldsymbol{\omega}}t}\boldsymbol{q}\left(0\right) \tag{3.2}$$

where $\boldsymbol{q}\left(0\right)$ is the coordinates of the point at time $t = 0$.
Note that $e^{\hat{\boldsymbol{\omega}}t}$ is the matrix exponential and has the form:

$$e^{\hat{\boldsymbol{\omega}}t} = \mathbb{I}_3 + \frac{\left(\hat{\boldsymbol{\omega}}t\right)}{1!} + \frac{\left(\hat{\boldsymbol{\omega}}t\right)^2}{2!} + \frac{\left(\hat{\boldsymbol{\omega}}t\right)^3}{3!} + \dots \tag{3.3}$$

If we rotate about our axis for θ units of time, then our point $\boldsymbol{q}\left(0\right)$ is transformed by the matrix exponent $e^{\hat{\boldsymbol{\omega}}\theta}$. This gives us the expression:

$$\boldsymbol{q}\left(\theta\right) = e^{\hat{\boldsymbol{\omega}}\theta}\boldsymbol{q}\left(0\right)$$

We see that $e^{\hat{\boldsymbol{\omega}}\theta}$ is a Rotation matrix defined by ω and θ, and rotates a point $\boldsymbol{q}\left(0\right)$ to the new position $\boldsymbol{q}\left(\theta\right)$. In this manner we define the exponential form of a rotation by:

$$R\left(\boldsymbol{\omega}, \theta\right) = e^{\hat{\boldsymbol{\omega}}\theta} \tag{3.4}$$

Using Equation 3.3, we can write this rotation as:

$$R\left(\boldsymbol{\omega}, \theta\right) = \mathbb{I}_3 + \frac{\theta}{1!}\hat{\boldsymbol{\omega}} + \frac{\theta^2}{2!}\hat{\boldsymbol{\omega}}^2 + \frac{\theta^3}{3!}\hat{\boldsymbol{\omega}}^3 + \dots$$

This can be simplified by noting that $\hat{\boldsymbol{\omega}}$ is a skew symmetric matrix and therefore has the following properties:

$$\hat{\boldsymbol{\omega}}^2 = \boldsymbol{\omega}\boldsymbol{\omega}^T - \left\|\boldsymbol{\omega}\right\|^2 \mathbb{I}_3$$
$$\hat{\boldsymbol{\omega}}^3 = -\left\|\boldsymbol{\omega}\right\|^2 \hat{\boldsymbol{\omega}}$$

We can therefore simplify Equation 3.2 to:

$$R\left(\boldsymbol{\omega}, \theta\right) = \mathbb{I}_3 + \frac{\theta}{1!}\hat{\boldsymbol{\omega}} + \frac{\theta^2}{2!}\hat{\boldsymbol{\omega}}^2 - \left\|\boldsymbol{\omega}\right\|^2 \frac{\theta^3}{3!}\hat{\boldsymbol{\omega}} - \left\|\boldsymbol{\omega}\right\|^2 \frac{\theta^4}{4!}\hat{\boldsymbol{\omega}}^2 + \dots$$

$$R\left(\boldsymbol{\omega}, \theta\right) = \mathbb{I}_3 + \left(\frac{\theta}{1!} - \left\|\boldsymbol{\omega}\right\|^2 \frac{\theta^3}{3!} + \dots\right)\hat{\boldsymbol{\omega}} + \left(\frac{\theta^2}{2!} - \left\|\boldsymbol{\omega}\right\|^2 \frac{\theta^4}{4!} + \dots\right)\hat{\boldsymbol{\omega}}^2$$

$$R\left(\boldsymbol{\omega}, \theta\right) = \mathbb{I}_3 + \left(\frac{\theta}{1!} - \left\|\boldsymbol{\omega}\right\|^3 \frac{\theta^3}{3!} + \dots\right)\frac{\hat{\boldsymbol{\omega}}}{\left\|\boldsymbol{\omega}\right\|} + \left(\left\|\boldsymbol{\omega}\right\|^2 \frac{\theta^2}{2!} - \left\|\boldsymbol{\omega}\right\|^4 \frac{\theta^4}{4!} + \dots\right)\frac{\hat{\boldsymbol{\omega}}^2}{\left\|\boldsymbol{\omega}\right\|^2}$$

Using the trigonometric identities:

$$sin\left(\left\|\boldsymbol{\omega}\right\|\theta\right) = \left\|\boldsymbol{\omega}\right\|\theta - \frac{\left(\left\|\boldsymbol{\omega}\right\|\theta\right)^3}{3!} + \frac{\left(\left\|\boldsymbol{\omega}\right\|\theta\right)^5}{5!} - \dots$$

$$cos\left(\|\boldsymbol{\omega}\|\,\theta\right) = 1 - \frac{\left(\|\boldsymbol{\omega}\|\,\theta\right)^2}{2!} + \frac{\left(\|\boldsymbol{\omega}\|\,\theta\right)^4}{4!} - \dots$$

we can write

$$R\left(\boldsymbol{\omega}, \theta\right) = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} sin\left(\|\boldsymbol{\omega}\|\,\theta\right) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}\left(1 - cos\left(\|\boldsymbol{\omega}\|\,\theta\right)\right)$$

which is Rodrigues' formula.
Now that we have an exponential relation on for the space of Rotations, $\mathcal{SO}\left(3\right)$, we now look for an extension to the space of rotations and translations, the space of rigid body motion $\mathcal{SE}\left(3\right)$.

3.3 Exponential Formulation for Rigid Body motion

Rigid body motion can be expressed as a combination of both a rotation and a translation. We find our exponential formulation for rigid body motion by deriving an expression for pure rotation and pure translation.

Pure Affine Rotation

We extend our previous derivation by allowing the axis of rotation to be affine- it no longer has to pass through the origin. Consider the rotation of a point $\boldsymbol{p}$ about an axis $\boldsymbol{\omega}$. The point $\boldsymbol{q}$ is a point on the axis itself. If we assume that the rotation occurs at unit velocity, we can write velocity of point $\boldsymbol{p}$ as:

$$\dot{\boldsymbol{p}}\left(t\right) = \boldsymbol{\omega} \times \left(\boldsymbol{p}\left(t\right) - \boldsymbol{q}\right) \tag{3.5}$$

This can be rewritten as the homogeneous linear equation:

$$\begin{bmatrix} \dot{\boldsymbol{p}}\left(t\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}\left(t\right) \\ 1 \end{bmatrix}$$

We can simplify this expression by setting:

$$\boldsymbol{v} = -\boldsymbol{\omega} \times \boldsymbol{q} \qquad \hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \tag{3.6}$$

giving us:

$$\dot{\bar{\boldsymbol{p}}} = \hat{\boldsymbol{\xi}}\bar{\boldsymbol{p}} \tag{3.7}$$

The solution of this differential equation is given by:

$$\bar{\boldsymbol{p}}\left(t\right) = e^{\widehat{\boldsymbol{\xi}t}}\bar{\boldsymbol{p}}\left(0\right)$$

Where $e^{\widehat{\boldsymbol{\xi}t}}$ is the matrix exponential of the 4×4 matrix $\widehat{\boldsymbol{\xi}t}$. This can be written as:

$$e^{\widehat{\boldsymbol{\xi}t}} = \mathbb{I}_4 + \frac{\left(\hat{\boldsymbol{\xi}}t\right)^1}{1!} + \frac{\left(\hat{\boldsymbol{\xi}}t\right)^2}{2!} + \frac{\left(\hat{\boldsymbol{\xi}}t\right)^3}{3!} + \frac{\left(\hat{\boldsymbol{\xi}}t\right)^4}{4!} + \dots \tag{3.8}$$

If we take θ to be the total rotation about the axis then we get the find that the transform for the rotational component is given by:

$$\bar{\boldsymbol{p}}\left(\theta\right) = e^{\widehat{\boldsymbol{\xi}\theta}}\bar{\boldsymbol{p}}\left(0\right) \tag{3.9}$$

Pure Translation

We now look at the contribution of a pure translational movement. The velocity of a point $\boldsymbol{p}$ moving along an axis $\boldsymbol{v}$ is written as:

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{v} \tag{3.10}$$

The solution of this equation can therefore be written as:

$$\bar{\boldsymbol{p}}(t) = e^{\widehat{\boldsymbol{\xi}} t}\bar{\boldsymbol{p}}(0) \quad with \quad \widehat{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{0}_{3\times3} & \boldsymbol{v} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix} \tag{3.11}$$

General Rigid Body Motion

As we can see, the representation of Rotations and Translations take the form of:

$$\bar{\boldsymbol{p}}(\theta) = e^{\widehat{\boldsymbol{\xi}\theta}}\bar{\boldsymbol{p}}(0)$$

where

$$\widehat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

Given this form of $\widehat{\boldsymbol{\xi}}$, we define its vector representation and the 'vee' $\vee$ and 'wedge' $\wedge$ operators as:

$$\hat{\boldsymbol{\xi}}^{\vee} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{\xi}$$

and

$$\boldsymbol{\xi}^{\wedge} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} = \hat{\boldsymbol{\xi}}$$

We call this vector $\boldsymbol{\xi} \in \mathbb{R}^{6\times1}$ a *twist*. A twist can be used to generate a rigid body motion via the matrix exponent $e^{\widehat{\boldsymbol{\xi}\theta}}$.

Differences in Interpretation

It is important to note that there is a subtle difference between the rigid body transform given by $g = e^{\widehat{\boldsymbol{\xi}\theta}}$, and the ones created using R_{AB} and p_{AB}. The rigid body transform formed from R_{AB}, p_{AB} is a mapping between two coordinate frames. The rigid body motion found using $e^{\widehat{\boldsymbol{\xi}\theta}}$ however is mappings from an original configuration to another after a rigid motion is applied: $\bar{\boldsymbol{p}}(\theta) = e^{\widehat{\boldsymbol{\xi}\theta}}\bar{\boldsymbol{p}}(0)$

3.4 Relating Twists to Rigid Body Transforms

Given our twist vector, we wish to find an expression for rigid body motion. We look at two cases- when there is no rotation (i.e. $\boldsymbol{\omega} = 0$) and when there is a rotation.

No Rotation $\boldsymbol{\omega} = 0$

If we have no rotation, then $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} & \mathbf{0}_{1\times 3} \end{bmatrix}^T$. We know that the transformation from an initial point to a final point is given by Equation 3.9. Using our equation for the matrix exponential (Equation 3.8, page 13) can be used to simplify this expression. We can write $\hat{\boldsymbol{\xi}}$ as :

Equation 3.9: $\bar{\boldsymbol{p}}(\theta) = e^{\widehat{\boldsymbol{\xi}\theta}}$

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \boldsymbol{v}^T \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}$$

We can see that all the higher powers of $\widehat{\boldsymbol{\xi}}$ will be zero. Therefore, we can write $e^{\widehat{\boldsymbol{\xi}\theta}}$ as:

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \mathbb{I}_3 + \frac{1}{1!}\left(\widehat{\boldsymbol{\xi}\theta}\right) + \frac{1}{2!}\left(\widehat{\boldsymbol{\xi}\theta}\right)^2 + \frac{1}{3!}\left(\widehat{\boldsymbol{\xi}\theta}\right)^3 + \dots$$

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 0 \end{bmatrix}$$

Consider the translation $\boldsymbol{p}$. If we take $\boldsymbol{v} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$ and $\theta = \|\boldsymbol{p}\|$ then:

$$\boldsymbol{\xi} = \begin{bmatrix} \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} \\ \mathbf{0} \end{bmatrix} \qquad \widehat{\boldsymbol{\xi}\theta} = \begin{bmatrix} \mathbf{0}_3 & \boldsymbol{p} \\ \mathbf{0} & 0 \end{bmatrix} \quad e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{p} \\ \mathbf{0} & 0 \end{bmatrix}$$

where $e^{\widehat{\boldsymbol{\xi}\theta}}$ is our expected rigid body representation of a translation by $\boldsymbol{p}$.

With Rotation $\boldsymbol{\omega} \neq 0$

Consider the rigid body motion defined by rotation $\boldsymbol{\omega}$ and translation $\boldsymbol{v}$. For our derivation we assume that $\boldsymbol{\omega}$ is a unit vector, normalising if necessary. We wish to find an expression for $\boldsymbol{\xi}$ after our rigid body transformation. The new twist is denoted as $\boldsymbol{\xi}'$ and is written as: $\widehat{\boldsymbol{\xi}'} = \boldsymbol{g}^{-1}\widehat{\boldsymbol{\xi}}\boldsymbol{g}$ with the rigid body transform:

$$\boldsymbol{g} = \begin{bmatrix} \mathbb{I} & \boldsymbol{\omega} \times \boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}$$

We can therefore write the twist $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{bmatrix}^T$ as:

$$\widehat{\boldsymbol{\xi}'} = \begin{bmatrix} \mathbb{I} & -\boldsymbol{\omega} \times \boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{I} & \boldsymbol{\omega} \times \boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

We wish to find an expression for $e^{\widehat{\boldsymbol{\xi}\theta}}$. By using the relation:

$$e^{\widehat{\boldsymbol{\xi}\theta}} = e^{\boldsymbol{g}\left(\widehat{\boldsymbol{\xi}'\theta}\right)\boldsymbol{g}^{-1}} = \boldsymbol{g}e^{\widehat{\boldsymbol{\xi}'\theta}}\boldsymbol{g}^{-1} \tag{3.12}$$

we see that it is possible to find $e^{\widehat{\boldsymbol{\xi}\theta}}$ by finding an expression for $e^{\widehat{\boldsymbol{\xi}'\theta}}$. The higher order terms of the matrix exponent $e^{\widehat{\boldsymbol{\xi}'\theta}}$ can be simplified using:

$$\widehat{\boldsymbol{\omega}}\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = 0$$

which allows us to write:

$$\left(\widehat{\boldsymbol{\xi}'}\right)^2 = \begin{bmatrix} \boldsymbol{\omega}^2 & 0 \\ \mathbf{0} & 0 \end{bmatrix}, \qquad \left(\widehat{\boldsymbol{\xi}'}\right)^3 = \begin{bmatrix} \boldsymbol{\omega}^3 & 0 \\ \mathbf{0} & 0 \end{bmatrix}, \qquad \ldots$$

We can therefore write the matrix exponent $e^{\widehat{\boldsymbol{\xi}'\theta}}$as:

$$e^{\widehat{\boldsymbol{\xi}'\theta}} = \mathbb{I}_4 + \frac{1}{1!}\left(\widehat{\boldsymbol{\xi}\theta}\right) + \frac{1}{2!}\left(\widehat{\boldsymbol{\xi}\theta}\right)^2 + \frac{1}{3!}\left(\widehat{\boldsymbol{\xi}\theta}\right)^3 + \ldots$$

$$e^{\widehat{\boldsymbol{\xi}'\theta}} = \begin{bmatrix} \mathbb{I}_3 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \boldsymbol{\omega}^2 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} \boldsymbol{\omega}^3 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \ldots$$

$$e^{\widehat{\boldsymbol{\xi}'\theta}} = \begin{bmatrix} e^{\widehat{\boldsymbol{\omega}\theta}} & \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

Plugging this into Equation 3.12 gives us:

$$e^{\widehat{\boldsymbol{\xi}\theta}} = e^{\boldsymbol{g}\left(\widehat{\boldsymbol{\xi}'\theta}\right)\boldsymbol{g}^{-1}} = \boldsymbol{g}e^{\widehat{\boldsymbol{\xi}'\theta}}\boldsymbol{g}^{-1}$$

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} \mathbb{I} & \boldsymbol{\omega}\times\boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} e^{\widehat{\boldsymbol{\omega}\theta}} & \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbb{I} & -\boldsymbol{\omega}\times\boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} \mathbb{I} & \boldsymbol{\omega}\times\boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} e^{\widehat{\boldsymbol{\omega}\theta}} & e^{\widehat{\boldsymbol{\omega}\theta}}\left(-\boldsymbol{\omega}\times\boldsymbol{v}\right) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} e^{\widehat{\boldsymbol{\omega}\theta}} & e^{\widehat{\boldsymbol{\omega}\theta}}\left(-\boldsymbol{\omega}\times\boldsymbol{v}\right) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta + \boldsymbol{\omega}\times\boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$e^{\widehat{\boldsymbol{\xi}\theta}} = \begin{bmatrix} e^{\widehat{\boldsymbol{\omega}\theta}} & \left(\mathbb{I} - e^{\widehat{\boldsymbol{\omega}\theta}}\right)\left(\boldsymbol{\omega}\times\boldsymbol{v}\right) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

3.5 Twists and Screws

We now return to our concept of screws. In out twist notation we have two components $\boldsymbol{\omega}$ and $\boldsymbol{v}$ which correspond to the axis of rotation and velocity of a point. We would like to relate this to our concepts from Screw theory, pitch h, axis l and magnitude M.

Pitch h The pitch is defined as the distance along the axis between the loops of the helix. If $\boldsymbol{\omega} \neq 0$, it can be written as:

$$h = \frac{\boldsymbol{\omega}^T\boldsymbol{v}}{\|\boldsymbol{\omega}\|^2}$$

and $h = +\infty$ when $\boldsymbol{\omega} = 0$ It can be thought of as the projection of velocity vector $\boldsymbol{v}$ onto our rotational axis $\boldsymbol{\omega}$.

Axis l The rotational axis is defined by a direction vector and a point that this vector passes through. It can be written as:

$$l = \begin{cases} \left\{ \frac{\boldsymbol{\omega} \times \boldsymbol{v}}{\|\boldsymbol{\omega}\|^2} + \lambda \boldsymbol{\omega} : \lambda \in \mathbb{R} \right\} & if \quad \boldsymbol{\omega} \neq \mathbf{0} \\ \{0 + \lambda \boldsymbol{v} : \lambda \in \mathbb{R}\} & if \quad \boldsymbol{\omega} = \mathbf{0} \end{cases}$$

We see that the axis runs parallel to $\boldsymbol{\omega}$ for the rotational case, and passes through the point $\frac{\boldsymbol{\omega} \times \boldsymbol{v}}{\|\boldsymbol{\omega}\|^2}$. In the irrotational case, the line is parallel to vector $\boldsymbol{v}$ and passes through the origin.

Magnitude M The magnitude of the screw is the net rotational component (or translational if there is no rotation). It is represented by

$$M = \begin{cases} \|\boldsymbol{\omega}\| & if \quad \boldsymbol{\omega} \neq \mathbf{0} \\ \|\boldsymbol{v}\| & if \quad \boldsymbol{\omega} = \mathbf{0} \end{cases}$$

By choosing $\|\boldsymbol{\omega}\| = 1$ or ($\|\boldsymbol{v}\| = 1$), the magnitude of the twist $\widehat{\boldsymbol{\xi}\theta}$ is written $M = \theta$.

Chapter 4

Review

Congratulations. If you have made it this far into the reader/course, you have seen the different methods of computing rotations, rigidbody motion and Forward Kinematics. However, practice makes perfect. This chapter will cover a number of practice problems that will test your ability to work with these concepts. Answers are provided at the end.

4.1 Rotations

Single Rotations

1. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the X axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

2. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{3}$ radian rotation from the A coordinate frame to the B coordinate frame about the Y axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

3. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{2}$ radian rotation from the A coordinate frame to the B coordinate frame about the Z axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

4. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the X axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the A coordinate frame look in the B coordinate frame?

5. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the X axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

6. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the Y axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the A coordinate frame look in the B coordinate frame?

7. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the Y axis.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

8. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[1, 0, 0]^T$.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

9. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{3}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[0, 1, 0]^T$.

   b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

10. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{2}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[0, 0, 1]^T$.

    b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

11. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[0, 0, 1]^T$.

    b) How would a point represented by the vector $[1, 2, 3]^T$ in the A coordinate frame look in the B coordinate frame?

12. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector

$$\left[\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]^T$$

.

    b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

13. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[1, 1, 1]^T$.

    b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

14. a) Write the rotation matrix R_{AB} for a $\frac{\pi}{6}$ radian rotation from the A coordinate frame to the B coordinate frame about the axis given by the vector $[1, 1, 0]^T$.

    b) How would a point represented by the vector $[1, 2, 3]^T$ in the B coordinate frame look in the A coordinate frame?

Multiple Rotations

1. a) You are given the rotation matrices: R_{AB}, R_{BC}, R_{CD}. Write an expression for: R_{AD}.

2. a) You are given the rotation matrices: R_{AB}, R_{BC}, R_{AD}. Write an expression for: R_{CD}.

3. a) You are given the rotation matrices: R_{AB}, R_{BC}, R_{CD}. Write an expression for: R_{AA}.

4. a) You are given the Euler angles $[\theta_X, \theta_Y, \theta_Z]$ in $X-Y-Z$ notation. What is the corresponding rotation matrix after all these rotations are applied?

5. a) You are given the Euler angles $[\theta_X, \theta_Y, \theta_Z]$ in $Z-Y-X$ notation. What is the corresponding rotation matrix after all these rotations are applied?

6. a) You are given the Euler angles $[\theta_Z, \theta_Y, \phi_Z]$ in $Z_\theta - Y - Z_\phi$ notation. What is the corresponding rotation matrix after all these rotations are applied?

Valid Rotations

1. a) Is the following transformation a valid rotation matrix? If so, prove it, if not state why.

$$\boldsymbol{q}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_B$$

2. a) Is the following transformation a valid rotation matrix? If so, prove it, if not state why.

$$\boldsymbol{q}_A = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_B$$

3. a) Is the following transformation a valid rotation matrix? If so, prove it, if not state why.

$$\boldsymbol{q}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_B$$

4. a) Is the following transformation a valid rotation matrix? If so, prove it, if not state why.

$$\boldsymbol{q}_A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_B$$

5. a) Is the following transformation a valid rotation matrix? If so, prove it, if not state why.

$$\boldsymbol{q}_A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}_B$$

4.2 Rigid body Motion

Single Rigid Body Motion

1. a) Write the homogeneous representation g_{AB} for a rigid body motion comprising of a rotation about the X axis by $\frac{\pi}{6}$ radians at the point $\left[1, 2, 3\right]^T$.

   b) How would a the point with coordinates $\left[1, 2, 3\right]^T$ in the B coordinate frame be represented in the A coordinate frame?

2. a) Write the homogeneous representation g_{AB} for a rigid body motion comprising of a rotation about the Y axis by $\frac{\pi}{4}$ radians at the point $\left[1, 2, 3\right]^T$.

   b) How would a the point with coordinates $\left[1, 2, 3\right]^T$ in the B coordinate frame be represented in the A coordinate frame?

3. a) Write the homogeneous representation g_{AB} for a rigid body motion comprising of a rotation about the vector $\left[0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$ by $\frac{\pi}{6}$ radians at the point $\left[1, 2, 3\right]^T$.

   b) How would a the point with coordinates $\left[1, 2, 3\right]^T$ in the B coordinate frame be represented in the A coordinate frame?

4. a) Write the homogeneous representation g_{AB} for a rigid body motion comprising of a rotation about the Y axis by $\frac{\pi}{4}$ radians at the point $\left[1, 0, 0\right]^T$.

   b) How would a the point with coordinates $\left[1, 2, 3\right]^T$ in the A coordinate frame be represented in the B coordinate frame?

Multiple Rigid Body Motions

1. Consider the two rigid body motions comprising of:

   - A rotation of $\frac{\pi}{2}$ radians about the Z axis at the point $[1, 0, 0]^T$ in the A frame which we call $\boldsymbol{g}_{AB}$.
   - A rotation of $-\frac{\pi}{2}$ radians about the Z axis at the point $[1, 0, 0]^T$ in the B frame which we call $\boldsymbol{g}_{BC}$.

   a) Write an expression for the rigid body motion $\boldsymbol{g}_{AC}$.

   b) How would a point with the coordinates $[1, 2, 3]^T$ in the C coordinate frame be represented in the A frame?

   c) How would a point with the coordinates $[1, 2, 3]^T$ in the A coordinate frame be represented in the C frame?

2. Consider the two rigid body motions comprising of:

   - A rotation of $\frac{\pi}{2}$ radians about the Z axis at the point $[1, 0, 0]^T$ in the A frame which we call $\boldsymbol{g}_{AB}$.

- A rotation of $-\frac{\pi}{2}$ radians about the Z axis at the point $[1,0,0]^T$ in the B frame which we call $\boldsymbol{g}_{BC}$.

a) Write an expression for the rigid body motion $\boldsymbol{g}_{CA}$.

b) How would a point with the coordinates $[1,2,3]^T$ in the C coordinate frame be represented in the A frame?

c) How would a point with the coordinates $[1,2,3]^T$ in the A coordinate frame be represented in the C frame?

4.3 Forward Kinematics

1. Consider the robotic manipulator shown in Figure 4.1.

   a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.

   b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
   $$\boldsymbol{g}_{ST}(\theta_1,\ldots,\theta_n) = e^{\hat{\xi}_1\theta_1}\ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

   d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   e) The coordinates of a point $\boldsymbol{p}$ are $[1,2,3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x,t_y,t_z,l_0,l_1\} = \{5,1,2,2,5\}$ and the joint angles: $\{\theta_1\} = \left\{\frac{\pi}{4}\right\}$.

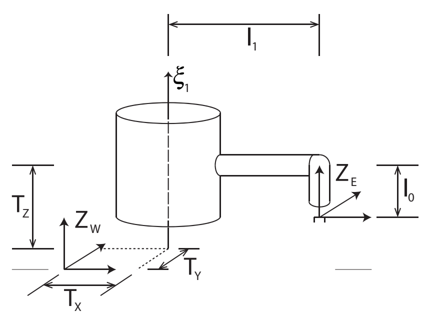

Figure 4.1: Single Axis Revolute Manipulator

2. *You will need MATLAB/Python to solve this question*
   Consider the robotic manipulator shown in Figure 4.2.

   a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.
   b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.
   c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
   $$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}\left(0\right)$$
   d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.
   e) The coordinates of a point $\boldsymbol{p}$ are $[1,2,3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x, t_y, t_z, l_0, l_1, \boldsymbol{\omega}_1\} = \left\{4, 5, 6, 2, 5, \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T\right\}$ and the joint angles: $\{\theta_1\} = \left\{\frac{\pi}{4}\right\}$.

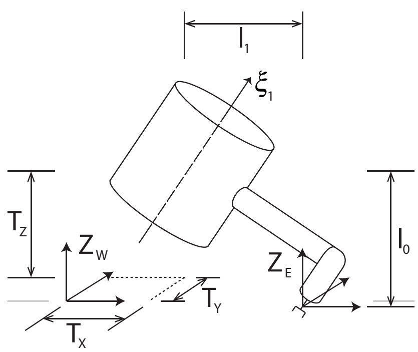

Figure 4.2: Single Off-Axis Revolute Manipulator

3. Consider the robotic manipulator shown in Figure 4.3.

   a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.
   b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.
   c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
   $$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}\left(0\right)$$

d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

e) The coordinates of a point $\boldsymbol{p}$ are $[1, 2, 3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x, t_y, t_z, l_0, l_1, l_2\} = \{2, 1, 2, 1, 3, 2\}$ and the joint angles: $\{\theta_1, \theta_2\} = \left\{\frac{\pi}{4}, -\frac{\pi}{4}\right\}$.

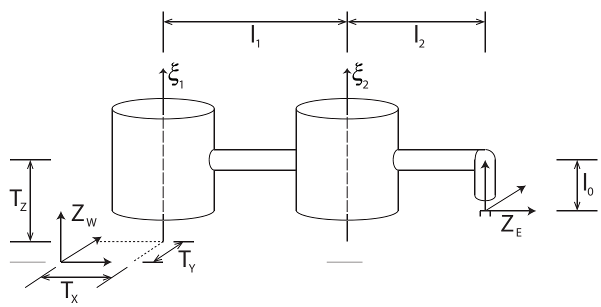

Figure 4.3: Double Axis Revolute Manipulator

4. Consider the robotic manipulator shown in Figure 4.4.

a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.

b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
$$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}\left(0\right)$$

d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

e) The coordinates of a point $\boldsymbol{p}$ are $[1, 2, 3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x, t_y, t_z, l_0, l_1, l_2, l_3\} = \{2, 1, 2, 1, 3, 2, 3\}$ and the joint angles: $\{\theta_1, \theta_2, \theta_3\} = \left\{\frac{\pi}{4}, -\frac{\pi}{8}, -\frac{\pi}{8}\right\}$.

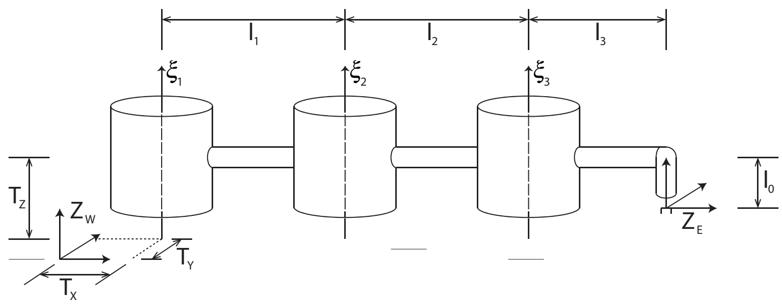

Figure 4.4: Triple Axis Revolute Manipulator

5. Consider the robotic manipulator shown in Figure 4.5.

   a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.

   b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
   $$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_n \theta_n} \boldsymbol{g}_{ST}\left(0\right)$$

   d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   e) The coordinates of a point $\boldsymbol{p}$ are $[1, 2, 3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x, t_y, t_z, l_0, l_1, l_2, l_3, l_4\} = \{2, 1, 2, 1, 3, 2, 3, 2\}$ and the joint angles: $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \left\{\frac{\pi}{4}, -\frac{\pi}{8}, -\frac{\pi}{8}, -1\right\}$.

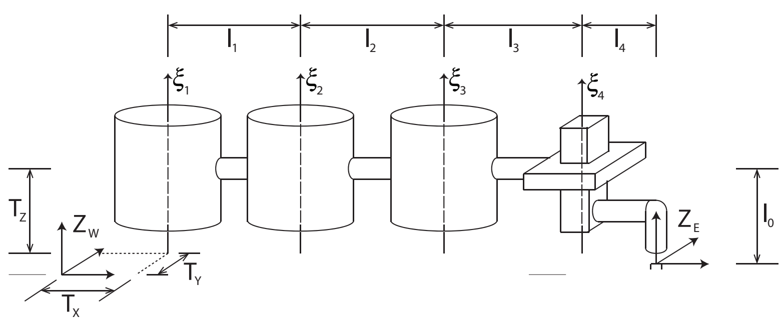

Figure 4.5: Triple Axis Revolute, Single Axis Prismatic Manipulator

6. Consider the robotic manipulator shown in Figure 4.6.

   a) Write the homogeneous transformations, for each joint of the manipulator. Each transformation should comprise of a rotation $\boldsymbol{R}$ and a translation $\boldsymbol{p}$.

   b) Using these homogeneous transformations, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   c) Write the twist $\boldsymbol{\xi}$ for each joint in the manipulator, using the following notation:
   $$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_n \theta_n} \boldsymbol{g}_{ST}\left(0\right)$$

   d) Using these twists, write an expression for the rigid body transformation that takes the coordinates of a point in the end effector frame and returns its coordinates in the world frame.

   e) The coordinates of a point $\boldsymbol{p}$ are $[1, 2, 3]^T$ in the end effector frame. Compute the coordinates of the point in the world frame, given the kinematic parameters: $\{t_x, t_y, t_z, l_0, l_1\} = \{2, 1, 2, 1, 3, \}$ and the joint angles: $\{\theta_1, \theta_2, \theta_3\} = \left\{\frac{\pi}{4}, -\frac{\pi}{8}, -\frac{\pi}{8}\right\}$.

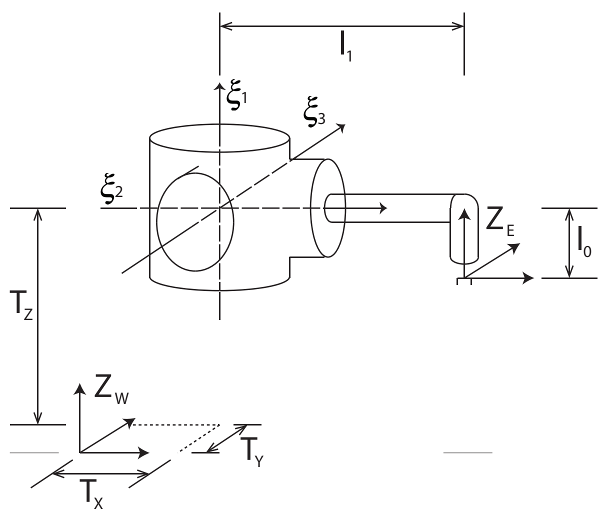

Figure 4.6: Triple Axis Revolute (Spherical) Manipulator

4.4 Answers

Single Rotations

1. a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8660 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0.2321 & 3.5981 \end{bmatrix}^T$$

2. a)

$$\begin{bmatrix} 0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & 0.5 \end{bmatrix}$$

b)

$$\begin{bmatrix} 3.0981 & 2 & 0.634 \end{bmatrix}^T$$

3. a)

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} -2 & 1 & 3 \end{bmatrix}^T$$

4. a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 3.2321 & 1.5981 \end{bmatrix}^T$$

5. a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8660 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0.2321 & 3.5981 \end{bmatrix}^T$$

6. a)

$$\begin{bmatrix} 0.8660 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.8660 \end{bmatrix}$$

b)

$$\begin{bmatrix}-0.6340 & 2 & 3.0981\end{bmatrix}^T$$

7. a)

$$\begin{bmatrix}0.8660 & 0 & 0.5\\ 0 & 1 & 0\\ -0.5 & 0 & 0.8660\end{bmatrix}$$

b)

$$\begin{bmatrix}2.3660 & 2 & 2.0981\end{bmatrix}^T$$

8. a)

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0.8660 & -0.5\\ 0 & 0.5 & 0.8660\end{bmatrix}$$

b)

$$\begin{bmatrix}1 & 0.2321 & 3.5981\end{bmatrix}^T$$

9. a)

$$\begin{bmatrix}0.5 & 0 & 0.8660\\ 0 & 1 & 0\\ -0.8660 & 0 & 0.5\end{bmatrix}$$

b)

$$\begin{bmatrix}3.0981 & 2 & 0.6340\end{bmatrix}^T$$

10. a)

$$\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

b)

$$\begin{bmatrix}-2 & 1 & 3\end{bmatrix}^T$$

11. a)

$$\begin{bmatrix}0.8660 & -0.5 & 0\\ 0.5 & 0.8660 & 0\\ 0 & 0 & 1\end{bmatrix}$$

b)

$$\begin{bmatrix}1.8660 & 1.2321 & 3\end{bmatrix}^T$$

12. a)

$$\begin{bmatrix}0.9107 & -0.2440 & 0.3333\\ 0.3333 & 0.9107 & -0.2440\\ -0.2440 & 0.3333 & 0.9107\end{bmatrix}$$

b)

$$\begin{bmatrix}1.4226 & 1.4226 & 3.1547\end{bmatrix}^T$$

13. a)

$$\begin{bmatrix}0.7441 & -0.3268 & 0.5837\\ 0.5827 & 0.7441 & -0.3268\\ -0.3268 & 0.5827 & 0.7441\end{bmatrix}$$

b)

$$\begin{bmatrix}1.8385 & 1.0906 & 3.0709\end{bmatrix}^T$$

14. a)

$$\begin{bmatrix}0.8691 & 0.1309 & 0.4770\\ 0.1309 & 0.8691 & -0.4770\\ -0.4770 & 0.4770 & 0.7381\end{bmatrix}$$

b)

$$\begin{bmatrix}2.5621 & 0.4379 & 2.6915\end{bmatrix}^T$$

Multiple Rotations

1. $\boldsymbol{R}_{AD} = \boldsymbol{R}_{AB}\boldsymbol{R}_{BC}\boldsymbol{R}_{CD}$
2. $\boldsymbol{R}_{CD} = [\boldsymbol{R}_{AB}\boldsymbol{R}_{BC}]^{-1}\,\boldsymbol{R}_{AD}$
3. $\boldsymbol{R}_{AA} = \mathbb{I}_3$
4. $\boldsymbol{R}_{XYZ} = \boldsymbol{R}_X\left(\theta_X\right)\boldsymbol{R}_Y\left(\theta_Y\right)\boldsymbol{R}_Z\left(\theta_Z\right)$
5. $\boldsymbol{R}_{ZYX} = \boldsymbol{R}_Z\left(\theta_Z\right)\boldsymbol{R}_Y\left(\theta_Y\right)\boldsymbol{R}_X\left(\theta_X\right)$
6. $\boldsymbol{R}_{ZYZ} = \boldsymbol{R}_Z\left(\theta_Z\right)\boldsymbol{R}_Y\left(\theta_Y\right)\boldsymbol{R}_Z\left(\phi_X\right)$

Valid Rotations

1. Valid rotation: Check Orthonormality, Orthogonality
2. Valid rotation: Check Orthonormality, Orthogonality
3. Not a valid Rotation: Determinant is 2, not orthonormal.
4. Valid rotation: Check Orthonormality, Orthogonality
5. Not a valid Rotation: Not skew symmetric, not orthonormal.

Single Rigid Body Motion

1. a)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0.8660 & -0.5 & 2 \\ 0 & 0.5 & 0.8660 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   b)

$$\begin{bmatrix} 2 & 2.2321 & 6.5981 \end{bmatrix}^T$$

2. a)

$$\begin{bmatrix} 0.7071 & 0 & 0.7071 & 1 \\ 0 & 1 & 0 & 2 \\ -0.7071 & 0 & 0.7071 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   b)

$$\begin{bmatrix} 3.8284 & 4 & 4.4142 \end{bmatrix}^T$$

3. a)

$$\begin{bmatrix} 0.8660 & -0.3536 & 0.3536 & 1 \\ 0.3536 & 0.9330 & 0.0670 & 2 \\ -0.3536 & 0.0670 & 0.9330 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   b)

$$\begin{bmatrix} 2.2196 & 4.4205 & 5.5795 \end{bmatrix}^T$$

4. a)

$$\begin{bmatrix} 0.7071 & 0 & 0.7071 & 1 \\ 0 & 1 & 0 & 0 \\ -0.7071 & 0 & 0.7071 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   b)

$$\begin{bmatrix} -2.1213 & 2 & 2.1213 \end{bmatrix}^T$$

Multiple Rigid Body Motion

1. a)

$$\boldsymbol{g}_{AC} = \boldsymbol{g}_{AB}\boldsymbol{g}_{BC}$$

$$\boldsymbol{g}_{AC} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix}2 & 3 & 3\end{bmatrix}^T$$

c)

$$\begin{bmatrix}0 & 1 & 3\end{bmatrix}^T$$

2. a)

$$\boldsymbol{g}_{CA} = \boldsymbol{g}_{CB}\boldsymbol{g}_{BA} = \boldsymbol{g}_{BC}^{-1}\boldsymbol{g}_{AB}^{-1}$$

$$\boldsymbol{g}_{CA} = \begin{bmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

b)

$$\begin{bmatrix}2 & 3 & 3\end{bmatrix}^T$$

c)

$$\begin{bmatrix}0 & 1 & 3\end{bmatrix}^T$$

Forward Kinematics

It should be possible to solve all these problems only using a calculator (except if stated otherwise).

1. a)

$$\boldsymbol{g}_{W1}(\theta_1) = \begin{bmatrix}cos(\theta_1) & -sin(\theta_1) & 0 & T_x\\ sin(\theta_1) & cos(\theta_1) & 0 & T_y\\ 0 & 0 & 1 & T_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

b)

$$\boldsymbol{g}_{WE} = \boldsymbol{g}_{W1}\boldsymbol{g}_{1E}$$

$$\boldsymbol{g}_{WE} = \begin{bmatrix}cos(\theta_1) & -sin(\theta_1) & 0 & T_x\\ sin(\theta_1) & cos(\theta_1) & 0 & T_y\\ 0 & 0 & 1 & T_z\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & l_1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{g}_{WE} = \begin{bmatrix}cos(\theta_1) & -sin(\theta_1) & 0 & T_x + l_1 cos(\theta_1)\\ sin(\theta_1) & cos(\theta_1) & 0 & T_y + l_1 sin(\theta_1)\\ 0 & 0 & 1 & T_z - l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1 = \begin{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}T_y\\ -T_x\\ 0\\ 0\\ 0\\ 1\end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}(\theta_1) = e^{\hat{\boldsymbol{\xi}}_1\theta_1}\boldsymbol{g}_{WE}(0)$$

$$\boldsymbol{g}_{WE}(\theta_1) = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}_1\theta_1} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}_1\theta_1}\right)(\boldsymbol{\omega}_1 \times \boldsymbol{v}_1) + \boldsymbol{\omega}_1\boldsymbol{\omega}_1^T\boldsymbol{v}_1\theta_1 \\ \boldsymbol{0} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & T_x + l_1 \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z - l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{WE}(\theta_1) = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 & T_x + l_1 cos(\theta_1) \\ sin(\theta_1) & cos(\theta_1) & 0 & T_y + l_1 sin(\theta_1) \\ 0 & 0 & 1 & T_z - l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$\begin{bmatrix} 7.8284 & 6.6569 & 3 \end{bmatrix}^T$$

2. a)

$$\boldsymbol{g}_{W1}(\theta_1) = \begin{bmatrix} \boldsymbol{R}_{W1} & \boldsymbol{p}_{W1} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

where

$$\boldsymbol{R}_{W1} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}(sin\|\boldsymbol{\omega}\|(\theta_1)) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1 - cos\|\boldsymbol{\omega}\|(\theta_1))$$

$$\boldsymbol{p}_{W1} = \begin{bmatrix} T_x & T_y & T_z \end{bmatrix}^T$$

b)

$$\boldsymbol{g}_{WE} = \boldsymbol{g}_{W1}\boldsymbol{g}_{1E}$$

$$\boldsymbol{g}_{WE} = \begin{bmatrix} \boldsymbol{R}_{W1} & \boldsymbol{p}_{W1} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{W1} = \begin{bmatrix} \boldsymbol{R}_{W1} & \boldsymbol{R}_{W1}\begin{bmatrix} l_1 \\ 0 \\ -l_0 \end{bmatrix} + \boldsymbol{p}_{W1} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -\boldsymbol{\omega} \times \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \\ \boldsymbol{\omega} \end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}(\theta_1) = e^{\hat{\boldsymbol{\xi}}_1\theta_1}\boldsymbol{g}_{WE}(0)$$

$$\boldsymbol{g}_{WE}(\theta_1) = e^{\hat{\boldsymbol{\xi}}_1\theta_1}\begin{bmatrix}1 & 0 & 0 & T_x + l_1\\ 0 & 1 & 0 & T_y\\ 0 & 0 & 1 & T_z - l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

e)

$$\begin{bmatrix}9.9142 & 7.0858 & 4.7071\end{bmatrix}^T$$

3. a)

$$\boldsymbol{g}_{W1}(\theta_1) = \begin{bmatrix}cos(\theta_1) & -sin(\theta_1) & 0 & T_x\\ sin(\theta_1) & cos(\theta_1) & 0 & T_y\\ 0 & 0 & 1 & T_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{g}_{12}(\theta_2) = \begin{bmatrix}cos(\theta_2) & -sin(\theta_2) & 0 & l_1\\ sin(\theta_2) & cos(\theta_2) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

b)

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2) = \boldsymbol{g}_{W1}(\theta_1)\,\boldsymbol{g}_{12}(\theta_2)\,\boldsymbol{g}_{2E}$$

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2) = \boldsymbol{g}_{W1}(\theta_1)\,\boldsymbol{g}_{12}(\theta_2)\begin{bmatrix}1 & 0 & 0 & l_2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1 = \begin{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}T_y\\-T_x\\0\\0\\0\\1\end{bmatrix}$$

$$\boldsymbol{\xi}_2 = \begin{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}T_x + l_1\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}T_y\\-T_x - l_1\\0\\0\\0\\1\end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2) = e^{\hat{\boldsymbol{\xi}}_1\theta_1}e^{\hat{\boldsymbol{\xi}}_{s_2}\theta_2}\boldsymbol{g}_{WE}(0)$$

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2) = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} \begin{bmatrix} 1 & 0 & 0 & T_x + l_1 + l_2 \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z - l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$\begin{bmatrix} 7.1213 & 5.1213 & 4 \end{bmatrix}^T$$

4. a)

$$\boldsymbol{g}_{W1}(\theta_1) = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 & T_x \\ sin(\theta_1) & cos(\theta_1) & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{12}(\theta_2) = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 & l_1 \\ sin(\theta_2) & cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{23}(\theta_3) = \begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & l_2 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2,\theta_3) = \boldsymbol{g}_{W1}(\theta_1)\,\boldsymbol{g}_{12}(\theta_2)\,\boldsymbol{g}_{23}(\theta_3)\,\boldsymbol{g}_{3E}$$

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2,\theta_3) = \boldsymbol{g}_{W1}(\theta_1)\,\boldsymbol{g}_{12}(\theta_2)\,\boldsymbol{g}_{23}(\theta_3) \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\xi}_2 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x + l_1\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} T_y \\ -T_x - l_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\xi}_3 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x + l_1 + l_2\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} T_y \\ -T_x - l_1 - l_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3\right) = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} e^{\hat{\boldsymbol{\xi}}_2 \theta_2} e^{\hat{\boldsymbol{\xi}}_3 \theta_3} \boldsymbol{g}_{WE}\left(0\right)$$

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3\right) = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} e^{\hat{\boldsymbol{\xi}}_2 \theta_2} e^{\hat{\boldsymbol{\xi}}_3 \theta_3} \begin{bmatrix} 1 & 0 & 0 & T_x + l_1 + l_2 + l_3 \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z - l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$\begin{bmatrix} 9.9691 & 5.8867 & 4 \end{bmatrix}^T$$

5. a)

$$\boldsymbol{g}_{W1}\left(\theta_1\right) = \begin{bmatrix} cos\left(\theta_1\right) & -sin\left(\theta_1\right) & 0 & T_x \\ sin\left(\theta_1\right) & cos\left(\theta_1\right) & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{12}\left(\theta_2\right) = \begin{bmatrix} cos\left(\theta_2\right) & -sin\left(\theta_2\right) & 0 & l_1 \\ sin\left(\theta_2\right) & cos\left(\theta_2\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{23}\left(\theta_3\right) = \begin{bmatrix} cos\left(\theta_3\right) & -sin\left(\theta_3\right) & 0 & l_2 \\ sin\left(\theta_3\right) & cos\left(\theta_3\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_{34}\left(\theta_4\right) = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3, \theta_4\right) = \boldsymbol{g}_{W1}\left(\theta_1\right) \boldsymbol{g}_{12}\left(\theta_2\right) \boldsymbol{g}_{23}\left(\theta_3\right) \boldsymbol{g}_{34}\left(\theta_4\right) \boldsymbol{g}_{4E}$$

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3, \theta_4\right) = \boldsymbol{g}_{W1}\left(\theta_1\right) \boldsymbol{g}_{12}\left(\theta_2\right) \boldsymbol{g}_{23}\left(\theta_3\right) \boldsymbol{g}_{34}\left(\theta_4\right) \begin{bmatrix} 1 & 0 & 0 & l_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}T_y\\-T_x\\0\\0\\0\\1\end{bmatrix}$$

$$\boldsymbol{\xi}_2 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x+l_1\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}T_y\\-T_x-l_1\\0\\0\\0\\1\end{bmatrix}$$

$$\boldsymbol{\xi}_3 = \begin{bmatrix} -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}T_x+l_1+l_2\\T_y\\T_z\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}T_y\\-T_x-l_1-l_2\\0\\0\\0\\1\end{bmatrix}$$

$$\boldsymbol{\xi}_4 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2,\theta_3,\theta_4) = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4} \boldsymbol{g}_{WE}(0)$$

$$\boldsymbol{g}_{WE}(\theta_1,\theta_2,\theta_3,\theta_4) = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4} \begin{bmatrix}1 & 0 & 0 & T_x+l_1+l_2+l_3+l_4\\ 0 & 1 & 0 & T_y\\ 0 & 0 & 1 & T_z-l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

e)

$$\begin{bmatrix}11.9691 & 5.8867 & 3\end{bmatrix}^T$$

6. a)

$$\boldsymbol{g}_{W1}(\theta_1) = \begin{bmatrix}cos(\theta_1) & -sin(\theta_1) & 0 & T_x\\ sin(\theta_1) & cos(\theta_1) & 0 & T_y\\ 0 & 0 & 1 & T_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{g}_{12}(\theta_2) = \begin{bmatrix}cos(\theta_2) & 0 & sin(\theta_2) & 0\\ 0 & 1 & 0 & 0\\ -sin(\theta_2) & 0 & cos(\theta_2) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{g}_{23}\left(\theta_3\right)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & cos\left(\theta_3\right) & -sin\left(\theta_3\right) & 0 & 0\\ 0 & sin\left(\theta_3\right) & cos\left(\theta_3\right) & 0 & 0\\ 0 & 0 & 0 & 1 & \end{bmatrix}$$

b)

$$\boldsymbol{g}_{WE}\left(\theta_1,\theta_2\right)=\boldsymbol{g}_{W1}\left(\theta_1\right)\boldsymbol{g}_{12}\left(\theta_2\right)\boldsymbol{g}_{23}\left(\theta_3\right)\boldsymbol{g}_{3E}$$

$$\boldsymbol{g}_{WE}\left(\theta_1,\theta_2\right)=\boldsymbol{g}_{W1}\left(\theta_1\right)\boldsymbol{g}_{12}\left(\theta_2\right)\boldsymbol{g}_{23}\left(\theta_3\right)\begin{bmatrix}1 & 0 & 0 & l_1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

c)

$$\boldsymbol{\xi}_1=\begin{bmatrix}-\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}=\begin{bmatrix}T_y\\-T_x\\0\\0\\0\\1\end{bmatrix}$$

$$\boldsymbol{\xi}_2=\begin{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\times\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}1\\0\\0\end{bmatrix}\end{bmatrix}=\begin{bmatrix}0\\T_z\\-T_y\\1\\0\\0\end{bmatrix}$$

$$\boldsymbol{\xi}_3=\begin{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}\times\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}\\ \begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}=\begin{bmatrix}-T_z\\0\\T_x\\0\\1\\0\end{bmatrix}$$

d)

$$\boldsymbol{g}_{WE}\left(\theta_1,\theta_2,\theta_3\right)=e^{\hat{\boldsymbol{\xi}}_1\theta_1}e^{\hat{\boldsymbol{\xi}}_2\theta_2}e^{\hat{\boldsymbol{\xi}}_3\theta_3}e^{\hat{\boldsymbol{\xi}}_4\theta_4}\boldsymbol{g}_{WE}\left(0\right)$$

$$\boldsymbol{g}_{WE}\left(\theta_1,\theta_2,\theta_3\right)=e^{\hat{\boldsymbol{\xi}}_1\theta_1}e^{\hat{\boldsymbol{\xi}}_2\theta_2}e^{\hat{\boldsymbol{\xi}}_3\theta_3}e^{\hat{\boldsymbol{\xi}}_4\theta_4}\begin{bmatrix}1 & 0 & 0 & T_x+l_1\\ 0 & 1 & 0 & T_y\\ 0 & 0 & 1 & T_z-l_0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

e)

$$\begin{bmatrix}1.8512 & 5.2927 & 4.3560\end{bmatrix}^T$$

Chapter 5

Inverse Kinematics and Jacobians

Chapter 6

Forces, Wrenches and Inertial Properties

Chapter 7

Newtonian Dynamics

Chapter 8

Lagrangian Dynamics

Chapter 9

Review

Appendix A

Linear Algebra Review

There are a number of mathematical tools used in this course. This section acts as an informal summary of these concepts, aimed to act as a refresher on these topics.

A.1 Vectors

In this section we will cover the basics of vectors, vector spaces and defining a basis that can describe elements in these spaces.

Vector Spaces

A vector space over field $\mathcal{F}$ is defined by a set V of vectors $\boldsymbol{v}_i$ and two operations: addition $(+)$ and multiplication $(a\boldsymbol{v})$. Addition takes any two vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and assigns a third vector $\boldsymbol{v}_1 + \boldsymbol{v}_2$. Multiplication takes a scalar a and a vector $\boldsymbol{v}$ and assigns a new vector $a\boldsymbol{v}$. This is scalar multiplication. The vector space must also satisfy the following axioms:

Additive Associativity $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$.

Additive Commutativity $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.

Additive Identity There exists a vector $\mathbf{0} \in V$ called the *zero vector*, such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$.

Additive Inverse For every $\boldsymbol{u} \in V$ there exists a vector $-\boldsymbol{u} \in V$ called the *additive inverse of* $\boldsymbol{u}$, such that $\boldsymbol{u} + -\boldsymbol{u} = \mathbf{0}$.

Multiplicative Compatibility $a\,(b\boldsymbol{u}) = (ab)\,\boldsymbol{u}$.

Multiplicative Identity $\mathbb{1}\boldsymbol{u} = \boldsymbol{u}$ where $\mathbb{1}$ is the multiplicative identity in $\mathcal{F}$.

Multiplicative Distributivity with Vector addition $a\,(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$.

Multiplicative Distributivity with Field addition $(a + b)\,\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$.

Example A.1.1. $\mathbb{R}^2$ is a vector space over the field of reals, with additive identity $\begin{bmatrix}0\\0\end{bmatrix}$ and multiplicative identity $1 \in \mathbb{R}$.

Example A.1.2. $\mathbb{R}^3$ is a vector space over the field of reals, with additive identity $\begin{bmatrix}0\\0\\0\end{bmatrix}$ and multiplicative identity $1 \in \mathbb{R}$.

Vector Subspaces

Given a vector space $(V, \mathcal{F})$, a non-empty subset W of vector space V that is both closed under addition and scalar multiplication forms $(W, \mathcal{F})$ and is called a *subspace* of V. Subset W must contain the zero-vector of V (otherwise it is not a vector space).

Example A.1.3. $\mathbb{R}^2$ is not a vector subspace of $\mathbb{R}^3$ as elements of $\mathbb{R}^2$ are ordered pairs: $\begin{bmatrix}v_{1,1}\\v_{2,1}\end{bmatrix}$ while elements of $\mathbb{R}^3$ are ordered triples: $\begin{bmatrix}v_{1,1}\\v_{2,1}\\v_{3,1}\end{bmatrix}$. Similarly, $\mathbb{R}^3$ is not a vector subspace of $\mathbb{R}^2$.

Example A.1.4. The line of form $L = \begin{bmatrix}a\\0\end{bmatrix}$ where $a \in \mathbb{R}$ is a vector subspace of $\mathbb{R}^2$. L is a valid space and includes the additive identity when $a = 0$ and the multiplicative identity 1. Checking for additive closure, for any two points on the line $\boldsymbol{v}_1 = \begin{bmatrix}a_1\\0\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}a_2\\0\end{bmatrix}$,

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{bmatrix}a_1\\0\end{bmatrix} + \begin{bmatrix}a_2\\0\end{bmatrix} = \begin{bmatrix}a_1 + a_2\\0\end{bmatrix} \in L$$

and for multiplicative closure, for scalar b,

$$b\boldsymbol{v} = b\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}ba\\0\end{bmatrix} \in L$$

Therefore the line of the form $\begin{bmatrix}a\\0\end{bmatrix}$ where $a \in \mathbb{R}$ is a vector subspace of $\mathbb{R}^2$.

Vector Span

The span of a set of vectors $V_n = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_i, ..., \boldsymbol{v}_n\}$, where $V_n \subset V$, is the intersection of all subspaces containing V_n. Span V_n can also be thought of as the set of all finite linear combinations of $\boldsymbol{v}_i$ for $i \in [1, n]$. This can be written:

$$span\,(V_n) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i \middle| \lambda_i \in \mathcal{F}, \boldsymbol{v}_i \in V_n \right\}$$

Example A.1.5. The vectors $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\2\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ span $\mathbb{R}^2$. For any $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$, we seek to find λ_i for $i \in [1,3]$ that satisfy:

$$\begin{bmatrix}x\\y\end{bmatrix} = \lambda_1 \begin{bmatrix}1\\0\end{bmatrix} + \lambda_2 \begin{bmatrix}0\\2\end{bmatrix} + \lambda_3 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\lambda_1 + \lambda_3\\2\lambda_2 + \lambda_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & 2 & 1\end{bmatrix} \begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}$$

There are an infinite number of solutions to this set of two equations with three unknowns.

Linear Independence

A set of vectors V_n is linearly independent if:

$$\boldsymbol{v}_i \notin span\left(\{\boldsymbol{v}_1, ..., \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, ..., \boldsymbol{v}_n\}\right)$$

Similarly $\boldsymbol{v}_n$ is linearly dependant on $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}\}$ if:

$$\boldsymbol{v}_n = \sum_{i=1}^{n-1} \lambda_i \boldsymbol{v}_i$$

where $\lambda_i \in \mathcal{F}$.

Example A.1.6. Consider the vectors $\boldsymbol{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}0\\2\end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$
Any two vectors from $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly independent.
However all three vectors are not linearly independent. We can say that $\boldsymbol{v}_2$ is linearly dependant on $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ as:

$$\boldsymbol{v}_2 = \begin{bmatrix}0\\2\end{bmatrix} = \lambda_1 \begin{bmatrix}1\\0\end{bmatrix} + \lambda_3 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\lambda_1 + \lambda_3\\\lambda_3\end{bmatrix} = -2\boldsymbol{v}_1 + 2\boldsymbol{v}_3$$

Basis

A basis for a vector space is a linearly independent set of vectors that span the vector space. If a vector space has dimension n then basis *MUST* contain n vectors.

Example A.1.7. Consider the vectors $\boldsymbol{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}0\\2\end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$.
Any two vectors from $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ act as a basis for $\mathbb{R}^2$.
For example, if we choose $\boldsymbol{v}_1$, $\boldsymbol{v}_3$, we can say that for any vector $\boldsymbol{v} \in \mathbb{R}^2$:

$$\boldsymbol{v} = \begin{bmatrix}x\\y\end{bmatrix} = \lambda_1 \begin{bmatrix}1\\0\end{bmatrix} + \lambda_3 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \begin{bmatrix}\lambda_1\\\lambda_3\end{bmatrix}$$

$$\therefore \begin{bmatrix}\lambda_1\\\lambda_3\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{-1} \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x - y\\y\end{bmatrix}$$

Orthogonality and Orthonormality

Two vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are orthogonal if $\boldsymbol{v}_1^T, \boldsymbol{v}_2 = 0$. A set of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are orthogonal if $\boldsymbol{v}_i^T \boldsymbol{v}_j = 0$ for $i, j \in [1, n], i \neq j$. A set of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are orthonormal if

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & if \ i = j \\ 0 & otherwise \end{cases}$$

for $i, j \in [1, n]$.
Vectors can be normalised by using by dividing each vector by its length: $\hat{\boldsymbol{v}} = \left[\frac{1}{\|\hat{\boldsymbol{v}}\|_2} \hat{\boldsymbol{v}}\right]$
Given a set of vectors, we can find an orthonormal basis using the *Gram-Schmidt process*.

Example A.1.8. Consider the vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Only the subset of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is orthogonal. $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are not orthonormal. If we normalise $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ the resulting vectors $\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2$ are orthonormal:

$$\hat{\boldsymbol{v}}_1^T \hat{\boldsymbol{v}}_2 = \left[\frac{1}{\|\hat{\boldsymbol{v}}_1\|_2} \hat{\boldsymbol{v}}_1\right]^T \left[\frac{1}{\|\hat{\boldsymbol{v}}_2\|_2} \hat{\boldsymbol{v}}_2\right] = \left[\frac{1}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]^T \left[\frac{1}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\hat{\boldsymbol{v}}_1^T \hat{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \quad \hat{\boldsymbol{v}}_2^T \hat{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

A.2 Coordinate Systems

Given a vector $\boldsymbol{v}$ and a basis $\{\boldsymbol{b}_1, ..., \boldsymbol{b}_n\}$, we can write:

$$\boldsymbol{v} = \sum_{i=1}^{n} \lambda_i b_i$$

The vector $[\lambda_1, ..., \lambda_n]$ are the *coordinates* of vector $\boldsymbol{v}$. We can write:

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{b}_1 & \dots & \boldsymbol{b}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

Coordinate Transforms

The coordinates of a vector depend on the basis used. To convert the coordinate representation of a vector between two sets of basis we can use a *coordinate transform matrix*.

Consider the vector $\boldsymbol{v}$ written in basis $\begin{bmatrix} \boldsymbol{w}_1 & \dots & \boldsymbol{w}_n \end{bmatrix}$. Each $\boldsymbol{w}_i$ has a representation in the new basis $\begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_n \end{bmatrix}$.

We can write each basis vector in $\{\boldsymbol{w}\}$ as a linear combination in basis $\{\boldsymbol{u}\}$:

$$\boldsymbol{w}_j = \sum_{i=1}^{n} a_{i,j} \boldsymbol{u}_{\boldsymbol{i}} = \begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} a_{1,j} \\ \vdots \\ a_{n,j} \end{bmatrix}$$

we can therefore write:

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{w}_1 & \dots & \boldsymbol{w}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{n,1} \end{bmatrix} \end{bmatrix} & \dots & \begin{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} a_{1,n} \\ \vdots \\ a_{n,n} \end{bmatrix} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

We can therefore define the coordinate transform matrix A:

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = A \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

which can transform the coordinates $\boldsymbol{\lambda}$ in frame $\{\boldsymbol{w}_i\}$ into the coordinates $\boldsymbol{\mu}$ in frame $\{\boldsymbol{u}_i\}$.

Example A.2.1. Consider two sets of bases of $\mathbb{R}^2$:

$$\{\boldsymbol{w}_1, \boldsymbol{w}_2\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right\}, \quad \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

For any $\boldsymbol{v}$ in $\{\boldsymbol{w}_i\}$, we can write:

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$\boldsymbol{w}_1, \boldsymbol{w}_2$ can be written as:

$$\boldsymbol{w}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{1,2} \\ \mu_{2,2} \end{bmatrix}$$

We can therefore say that any vector $\boldsymbol{v}$ represented by coordinates $\boldsymbol{\lambda}$ in $\boldsymbol{w}_1, \boldsymbol{w}_2$ can be represented as:

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \mu_{1,1} & \mu_{1,2} \\ \mu_{2,1} & \mu_{2,2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} A\boldsymbol{\lambda}$$

where $A\boldsymbol{\lambda}$ is the representation in basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$
We double-check our result by looking at coordinates of the same point in each frame as shown in Figure A.1.

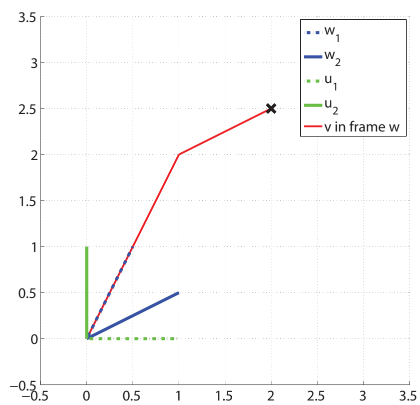

Figure A.1: We plot the vector represented by coordinates $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ in the $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ frame (shown in blue). Applying our coordinate transform matrix A, we obtain the coordinates $\left[\begin{smallmatrix} 2 \\ 2.5 \end{smallmatrix}\right]$, in frame $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ (shown in green). This agrees with our plot showing the construction of our vector in $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ (in red) and reading off its location from our axes.

Example A.2.2. Our second example looks at the effect a coordinate transform between two orthonormal bases. . Consider two sets of orthonormal bases of $\mathbb{R}^2$:

Check that these bases are orthonormal- the definition is shown on page 54

$$\{\boldsymbol{w}_1, \boldsymbol{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

We repeat the same process as in Example A.2.1, rewriting out basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ in $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$:

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \mu_{1,2} \\ \mu_{2,2} \end{bmatrix}$$

We then compute the transformation matrix:

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2\end{bmatrix} \begin{bmatrix}\mu_{1,1} & \mu_{1,2} \\ \mu_{2,1} & \mu_{2,2}\end{bmatrix} \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2\end{bmatrix} \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}$$

Therefore, the coordinates $\boldsymbol{\nu}$ in $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ can be computed from coordinates $\boldsymbol{\lambda}$ in $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ via:

$$\boldsymbol{\nu} = \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} \boldsymbol{\lambda}$$

By decomposing this orthonormal matrix into Euler angles, we see that this is a rotation by $-\frac{\pi}{4}$ radians: .

Euler angles to Rotation m in $\mathbb{R}^2$ is given by: $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$

$$\theta = atan2\left(\frac{\mu_{2,1}}{\mu_{2,2}}\right) = -\frac{\pi}{4}$$

If we plot these bases on a graph, we can see that basis $\{\boldsymbol{w}_i\}$ is indeed a clockwise rotation of $\frac{\pi}{4}$ radians from basis $\{\boldsymbol{u}_i\}$ as shown in Figure A.2.

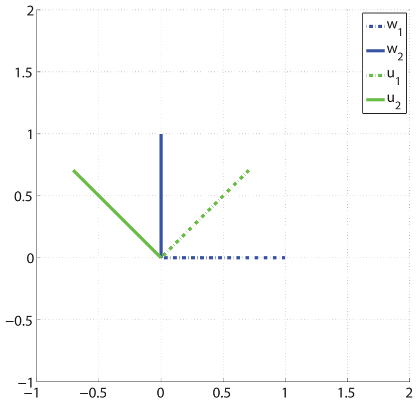

Figure A.2: Plot showing the two sets of bases, $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ in blue and $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ in green. We can see that the rotation from frame $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ in blue is $-\frac{\pi}{4}$ radians.

In general, the process of finding the coordinate transform matrix A can be simplified to:

$$A \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}^{-1} \begin{bmatrix}\boldsymbol{v}_1 & \dots & \boldsymbol{v}_n\end{bmatrix}$$

Proof. We have said that for each basis vector $\{\boldsymbol{w}_j\}$, we can write it as a linear combination of vectors from basis $\{\boldsymbol{u}_i\}$:

$$\boldsymbol{w}_j = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix} \boldsymbol{\mu}_j$$

For a given vector $\boldsymbol{v}$ with the coordinates $\boldsymbol{\lambda}$ in basis $\{\boldsymbol{w}_j\}$, we can write:

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{w}_1 & \dots & \boldsymbol{w}_n\end{bmatrix} \boldsymbol{\lambda}$$

Similarly, $\boldsymbol{v}$ can be written in basis $\{\boldsymbol{u}_i\}$ by the coordinates $\boldsymbol{\nu}$ as:

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix} \boldsymbol{\nu}$$

We want to find coordinate transform matrix A that will convert from $\boldsymbol{\lambda}$ in basis $\{\boldsymbol{w}_j\}$ to $\boldsymbol{\nu}$ in basis $\{\boldsymbol{u}_i\}$ via: $\boldsymbol{\nu} = A\boldsymbol{\lambda}$. From our expression for $\boldsymbol{v}$ in $\{\boldsymbol{w}_j\}$, we can say:

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{w}_1 & \dots & \boldsymbol{w}_n\end{bmatrix}\boldsymbol{\lambda}$$

$$\boldsymbol{v} = \begin{bmatrix}\begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\boldsymbol{\mu}_1 & \dots & \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\boldsymbol{\mu}_n\end{bmatrix}\boldsymbol{\lambda}$$

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{\mu}_1 & \dots & \boldsymbol{\mu}_n\end{bmatrix}\boldsymbol{\lambda} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\boldsymbol{\nu}$$

$$\therefore \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\boldsymbol{\nu} = \begin{bmatrix}\boldsymbol{w}_1 & \dots & \boldsymbol{w}_n\end{bmatrix}\boldsymbol{\lambda}$$

Pre-multiplying by the inverse of $\begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}$ gives:

$$\begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}\boldsymbol{\nu} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{w}_1 & \dots & \boldsymbol{w}_n\end{bmatrix}\boldsymbol{\lambda}$$

$$\mathbb{I}_{n\times n}\boldsymbol{\nu} = \begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_n\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{w}_1 & \dots & \boldsymbol{w}_n\end{bmatrix}\boldsymbol{\lambda}$$

Giving our expression for A. □

A.3 Matrix Partitioning

Consider the two matrices $M_1 \in \mathbb{R}^{m\times n}$ and $M_2 \in \mathbb{R}^{n\times p}$. These matrices are nothing but blocks of elements. We can subgroup elements in a matrix via *partitioning*.

$$\overset{m\times n}{[M_1]} = \begin{bmatrix}\overset{k\times l}{[A_1]} & \overset{k\times(n-l)}{[B_1]} \\ \overset{(m-k)\times l}{[C_1]} & \overset{(m-k)\times(n-l)}{[D_1]}\end{bmatrix} \quad \overset{n\times p}{[M_2]} = \begin{bmatrix}\overset{l\times r}{[A_2]} & \overset{l\times(p-r)}{[B_2]} \\ \overset{(n-l)\times r}{[C_2]} & \overset{(n-l)\times(p-r)}{[D_2]}\end{bmatrix}$$

We can perform matrix operations like matrix multiplication in this form:

$$\overset{m\times n}{[M_1]}\overset{n\times p}{[M_2]} = \begin{bmatrix}\overset{k\times r}{[[A_1]\,[A_2] + [B_1]\,[C_2]]} & \overset{k\times(p-r)}{[[A_1]\,[B_2] + [B_1]\,[D_2]]} \\ \overset{(m-k)\times r}{[[C_1]\,[A_2] + [D_1]\,[C_2]]} & \overset{(m-k)\times(p-r)}{[[C_1]\,[B_2] + [D_1]\,[D_2]]}\end{bmatrix}$$

Two important cases are when subgroup matrices into row and column matrices. Consider two matrices V and W which we subdivide into n column and row vectors respectively:

$$V = \begin{bmatrix}\boldsymbol{v}_1 & \dots & \boldsymbol{v}_n\end{bmatrix} \quad W = \begin{bmatrix}\boldsymbol{w}_1^T \\ \vdots \\ \boldsymbol{w}_n^T\end{bmatrix}$$

The matrix multiplication VW is:

$$VW = \begin{bmatrix}| \\ \boldsymbol{v}_1 \\ |\end{bmatrix}\begin{bmatrix}- & \boldsymbol{w}_1^T & -\end{bmatrix} + \cdots + \begin{bmatrix}| \\ \boldsymbol{v}_n \\ |\end{bmatrix}\begin{bmatrix}- & \boldsymbol{w}_n^T & -\end{bmatrix} = \sum_{i=1}^{n}\boldsymbol{v}_i\boldsymbol{w}_i^T$$

The matrix multiplication WV is:

$$WV = \begin{bmatrix}\boldsymbol{w}_1^T\boldsymbol{v}_1 & \dots & \boldsymbol{w}_1^T\boldsymbol{v}_n \\ \vdots & & \vdots \\ \boldsymbol{w}_n^T\boldsymbol{v}_1 & \dots & \boldsymbol{w}_n^T\boldsymbol{v}_n\end{bmatrix}$$

A.4 Linear Maps

Matrices can be thought of as Linear Maps. We introduce the key properties of Linear Maps, and their application to matrices.

Linearity

A map $f : X \to Y$ between the two vector spaces X, Y is linear if:

$$f(ax + by) = af(x) + bf(y)$$

where $x \in X$, $y \in Y$ and $a, b \in \mathbb{R}$.
Matrices are linear maps between vector spaces. We call the initial space that the linear map acts on the 'domain', and the final space that the linear map maps to the 'codomain'. The dimentions of these two spaces does not have to be the same.

Example A.4.1. Consider the linear map represented by the matrix A where:

$$\boldsymbol{y} = A\boldsymbol{x} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad with \quad \begin{matrix} \boldsymbol{x} \in X \\ \boldsymbol{y} \in Y \\ X = \mathbb{R}^3 \\ Y = \mathbb{R}^2 \end{matrix}$$

Here the domain is X and the codomain is Y. We see that A takes any vector in $\mathbb{R}^3$ and flattens it (by disregarding the x_3 component) into a vector in $\mathbb{R}^2$. If A was instead defined as its transpose $\left(A_{new} = A_{old}^T\right)$, it would take a vector from $\mathbb{R}^2$ and embed it into $\mathbb{R}^3$ as a plane.

If the matrix A is square, then the dimension of the two spaces X and Y will be the same.

Decomposition

Any matrix $A \in \mathbb{R}^{m \times n}$ decomposes its domain into two significant subspaces and its codomain into two significant subspaces. These subspaces are termed the *nullspace* and *rangespace*.

Nullspace The nullspace of a map f is the space of vectors in the domain that gets mapped onto zero. This is written as the set of vectors in X that satisfy

$$\{\boldsymbol{x} \in X | f(\boldsymbol{x}) = 0\}$$

.

Rangespace The rangespace of a map is the space of vectors in the codomain that f maps some vector $\boldsymbol{x} \in X$ to. This is written as the set of vectors in Y that satisfy

$$\{\boldsymbol{y} \in Y | \exists \boldsymbol{x} \in X \ s.t. \ f(\boldsymbol{x}) = \boldsymbol{y}\}$$

.

Using our definitions, we can determine how the range and null spaces our linear map relate to the matrix A.

Range of A The range of A denoted $\mathcal{R}(A)$ are the vectors $\boldsymbol{y} \in Y$ that satisfies $\{\boldsymbol{y} \in Y | \exists \boldsymbol{x} \in X s.t. A\boldsymbol{x} = \boldsymbol{y}\}$. Building on our intuition on bases and coordinate systems (Section A.2), we can see that the columns of A act as a basis for $\mathcal{R}(A)$.

$$\mathcal{R}(A) = \{\boldsymbol{y} \in Y | \exists \boldsymbol{x} \in X \ s.t. \ A\boldsymbol{x} = \boldsymbol{y}\}$$

$$\mathcal{R}(A) = \left\{ \boldsymbol{y} \in Y \middle| \exists \boldsymbol{x} \in X \ s.t. \ \begin{bmatrix} | & & | \\ \boldsymbol{c}_1 & \dots & \boldsymbol{c}_n \\ | & & | \end{bmatrix} \boldsymbol{x} = \boldsymbol{y} \right\}$$

Therefore the span of the columns of A is the range of A.

Nullspace of A The nullspace of A denoted $\mathcal{N}(A)$ are the vectors $\boldsymbol{x} \in X$ that satisfies $\{\boldsymbol{x} \in X | A\boldsymbol{x} = 0\}$. This can be thought of as the $\boldsymbol{x} \in X$ that are orthogonal to the rows of A.

$$\mathcal{N}(A) = \{\boldsymbol{x} \in X | A\boldsymbol{x} = 0\} = \left\{ \boldsymbol{x} \in X \middle| \begin{bmatrix} — & \boldsymbol{r}_1 & — \\ & \vdots & \\ — & \boldsymbol{r}_m & — \end{bmatrix} \boldsymbol{x} = 0 \right\}$$

It seems strange to say that the Nullspace of A as being orthogonal to the rows of A. By transposing matrix A, we can see that the rows of A are the columns of A^T. As we have transposed A, it is important to note that A^T can be thought of as a linear map from Y to X (rather than from X to Y). This means that the columns of A^T act as a basis for the vector space orthogonal to $\mathcal{N}(A)$. Using this observation we can make a statement about the nullspace and range of A^T.

Range of A^T As in the previous case, $\mathcal{R}\left(A^T\right)$ can be written as

$$\mathcal{R}\left(A^T\right) = \left\{\boldsymbol{x} \in X \middle| \exists \boldsymbol{y} \in Y \ s.t. \ A^T\boldsymbol{y} = \boldsymbol{x}\right\}$$

$$\mathcal{R}\left(A^T\right) = \left\{ \boldsymbol{x} \in X \middle| \exists \boldsymbol{y} \in Y \ s.t. \ \begin{bmatrix} | & & | \\ \boldsymbol{r}_1 & \dots & \boldsymbol{r}_n \\ | & & | \end{bmatrix} \boldsymbol{y} = \boldsymbol{x} \right\}$$

Hence $\mathcal{R}\left(A^T\right)$ is the span of the columns of A^T. We also know that $\mathcal{R}\left(A^T\right)$ is orthogonal to $\mathcal{N}(A)$. We can therefore say that the domain of A can be written as the direct sum of the two subspaces as shown:

$$dom(A) = \mathcal{N}(A) \oplus \mathcal{R}\left(A^T\right)$$

Nullspace of A^T Using the same method as before we can show that the nullspace of A^T is orthogonal to the rows of A^T, and that $\mathcal{N}\left(A^T\right)$ is orthogonal to $\mathcal{R}(A)$. We now have a decomposition for the codomain:

$$codom(A) = \mathcal{N}\left(A^T\right) \oplus \mathcal{R}(A)$$

Example A.4.2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ This matrix represents the linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$. From our definitions above, we see that:

$\mathcal{R}(A)$ This is the span of the columns of A. By inspection we see that this is the usual basis for $\mathbb{R}^2$.

$$\mathcal{R}(A) = \mathbb{R}^2$$

$\mathcal{N}(A)$ This is the subspace orthogonal to the rows of A. By inspection we see that this is the third component of a vector in $\mathbb{R}^3$.

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \right\} \forall a \in \mathbb{R}$$

$\mathcal{R}\left(A^T\right)$ This is the span of the columns of A^T. By inspection we see that this is a plane in $\mathbb{R}^3$ with the third component being kept at zero.

$$\mathcal{R}\left(A^T\right) = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right\} \forall a, b \in \mathbb{R}$$

$\mathcal{N}\left(A^T\right)$ This is the subspace orthogonal to the rows of A^T. By inspection we see that this can only be the zero vector.

$$\mathcal{N}\left(A^T\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We finally check our assertions on the domain and codomain:

$$\mathbb{R}^3 = dom\,(A) = \mathcal{N}(A) \oplus \mathcal{R}\left(A^T\right) = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right\} \forall a, b, c \in \mathbb{R}$$

$$\mathbb{R}^2 = codom\,(A) = \mathcal{N}\left(A^T\right) \oplus \mathcal{R}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \oplus \mathbb{R}^2$$

A.5 Square Matrices

In this course we will be primarily concerned with square matrices. There are a number of special parameters and properties that we will find useful.

Eigenvalue Problem

Given a matrix A, we seek to find a scalar λ and a vector $\boldsymbol{x}$ that satisfy:

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Here, λ is termed the *eigenvalue*, and vector $\boldsymbol{x}$ is the *eigenvector*.
In the special case where $\lambda = 0$, we can say that $\boldsymbol{x} \in \mathcal{N}(A)$.

Similarity Transforms

Given a linear transformation represented by matrix $A \in \mathbb{R}^{n\times n}$, where $A : X \to X$, if the basis used to represent coordinates changes, how should A change? Consider the coordinate change $\bar{X} = TX$. We wish to find $\bar{A}$ such that:

$$\bar{A}\bar{X} = TAX$$

$$\bar{A}\bar{X} = TAT^{-1}\bar{X}$$

This provides us with a *similarity transform* for A under the coordinate transform T:

$$\bar{A} = TAT^{-1}$$

A.6 Invertability of a Matrix

A square matrix A is said to be invertible if it has both a left and right inverse.

Left Inverse A matrix is said to have a left inverse if:

$$\exists A_L^{-1} \ s.t. \ A_L^{-1} A = I$$

Right Inverse A matrix is said to have a right inverse if:

$$\exists A_R^{-1} \ s.t. \ A A_R^{-1} = I$$

A matrix is invertible if there is a one-to-one correspondence between every element of the domain and codomain.

This is equivalent to saying that A is invertible if it is a *bijection*, being both *injective* and *surjective*.

In a practical sense it mean that elements in the domain and codomain are in pairs. For each pair of points $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$, we get the linking between them via:

$$\boldsymbol{y} = A\boldsymbol{x} \qquad \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

For a square matrix A the following statements are equivalent:

- A is invertible.
- A has full row rank (the only element in $\mathcal{N}(A)$ is 0.
- A has full column rank ($\mathcal{R}(A)$ spans the whole codomain.
- A has no zero eigenvalues.
- $det\,(A) \neq 0$.

the following statements are also equivalent:

- A is *not* invertible.
- A is singular.
- A has linearly dependant rows- A does not have full row rank. (This means there are nonzero elements in $\mathcal{N}(A)$, i.e. $\exists \boldsymbol{x} \neq 0 s.t. A\boldsymbol{x} = 0$)
- A has linearly dependant columns- A does not have full column rank. ($\mathcal{R}(A)$ does not span the whole codomain).
- A has at least one zero eigenvalue.
- $det\,(A) = 0$.

The intuition behind singular matrices, is that one or more dimensions of the domain are collapsed along the directions of the nullspace vectors.

Example A.6.1. Consider the linear mapping represented by matrix:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

This matrix does not have full column rank, column 3 is the sum of columns 1 and 2. We therefore know that this matrix is singular and therefore has a nullspace. The nullspace is characterised by the vectorspace orthogonal to the rows of A. We can find a basis for our nullspace by therefore taking the crossproduct of the rows of A.
The dimensional collapse of a singular matrix can be seen in figure A.3. As shown, points in or original $\mathbb{R}^3$ space can are collapsed along the Nullspace vector forming a plane.

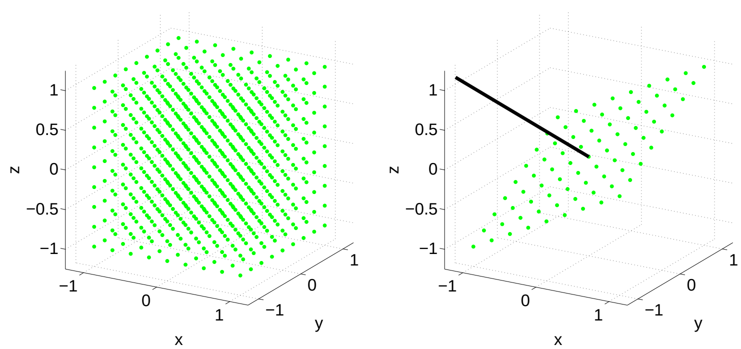

Figure A.3: The collapse of $\mathbb{R}^3$ using the linear mapping represented by matrix A. Left: Our original space with representative points shown in green. Right: the collapsed space (green) and the normal vector (black).

Pseudoinverse

An interesting extension to the inverse is the *Moore-Penrose Pseudoinverse*. Given any matrix A, an inverse-like matrix called the *pseudoinverse* denoted A^+ can be found. There are two definitions of the pseudo inverse depending on whether AA^T or AA^T is invertible.
If A^TA is invertible then the pseudoinverse can be written:

$$A^+ = \left(A^TA\right)^{-1} A^T$$

If AA^T is invertible then the pseudoinverse can be written:

$$A^+ = A^T \left(AA^T\right)^{-1}$$

The pseudoinverse effectively finds the least squares solution to a system of linear equations, is always defined and is unique. Example A.6.2 shows the utility of the pseudoinverse.

Example A.6.2. Consider our previous linear map represented by matrix:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

We showed in Example A.6.1 that this matrix was non-invertable. However let us use the pseudoinverse to flatten a volume of $\mathbb{R}^3$ via matrix A. Consider any vector $\boldsymbol{v}_X \in X$. This vector can be flattened to a corresponding point $\boldsymbol{v}_S \in S$ via:

$$\boldsymbol{v}_S = A\boldsymbol{v}_X$$

We can find a representation of this vector in our X space again by using the pseudoinverse. This gives us the projection of $\boldsymbol{v}_X$ onto the invertible part of A.

$$\boldsymbol{v}_{A^+S} = \boldsymbol{v}_{A^+AX} = A^+S$$

It is important to note that any vector that is parallel to nullspace vector and passes through this point will have the same representation.
Figure A.4 shows points of a unit cube in $\mathbb{R}^3$ before and after being flattened. After being flattened onto a $\mathbb{R}^2$ plane embedded in $\mathbb{R}^3$, all the points in this embedded plane now have a one-to-one relationship with the $\mathbb{R}^2$ in our $\boldsymbol{s}$, $\boldsymbol{t}$ basis.

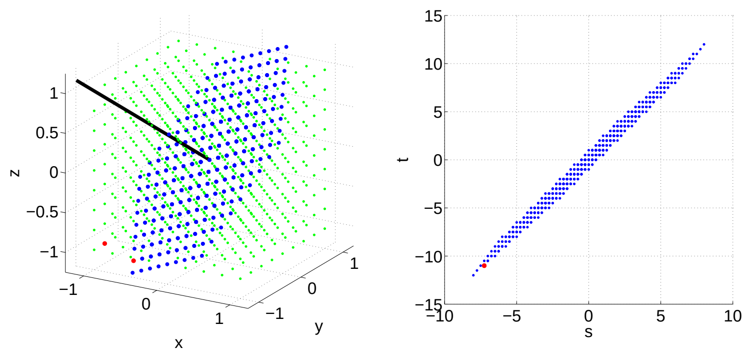

Figure A.4: The collapse of a unit cube in $\mathbb{R}^3$ using the linear mapping represented by matrix A. Left: Representation in $\mathbb{R}^3$ showing the original cube (green), its flattened representation (blue), the nullspace basis vector (black). Right: Our representation of the cube in $\mathbb{R}^2$ after mapping the points using the A matrix (blue). These are the points that are then used to give the flattened plane in $\mathbb{R}^3$. Given a point in $\mathbb{R}^3$ (shown in red), we track it through the different mappings. Its representation in $(\boldsymbol{s}, \boldsymbol{t})$ is shown in the right figure. If we use the pseudoinverse to find the representation of $(\boldsymbol{s}, \boldsymbol{t})$ in $\mathbb{R}^3$, we find it has been flattened onto the blue plane.

Appendix B

CheatSheet

This is the cheat sheet that will be provided for every midterm.

B.1 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & if\ i = j \\ 0 & otherwise \end{cases}$$

B.2 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

B.3 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} sin(\|\boldsymbol{\omega}\|\,\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1 - cos(\|\boldsymbol{\omega}\|\,\theta))$$

B.4 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

B.5 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1 \theta_1}$$

$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \boldsymbol{g}_{AB}(0)$$

$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1 \theta_1} \ldots e^{\hat{\xi}_n \theta_n} \boldsymbol{g}_{ST}(0)$$

Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}} \theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}} \theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}} \theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega} \boldsymbol{\omega}^T \boldsymbol{v} \theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}} \theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v} \theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$