Cramer's Rule

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$ invertible matrix

want to solve $A\vec{x} = \vec{b}$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i = \frac{\begin{vmatrix} | & & | & & | \\ \vec{a}_1 & \cdots & \vec{b} & \cdots & \vec{a}_n \\ | & & | & & | \end{vmatrix}}{\begin{vmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{vmatrix}}$$

replace i-th column of A

Ex) Use Cramer's rule to solve

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 4 & 7 & -1 \end{array}\right] \qquad x = \frac{\begin{vmatrix} 3 & 3 \\ -1 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}} = \frac{25}{2} = 12.5$$

$$y = \frac{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}}{2} = \frac{-14}{2} = -7$$

Ex] Use Cramer's rule to compute A^{-1} where $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \qquad AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} | & | \\ A\vec{v}_1 & A\vec{v}_2 \\ | & | \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\begin{vmatrix} 1 & 3 \\ 0 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}} = \frac{7}{2} \qquad \frac{\begin{vmatrix} 0 & 3 \\ 1 & 7 \end{vmatrix}}{2} = \frac{-3}{2}$$

$$\frac{\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}} = \frac{-4}{2} \qquad \frac{\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix}}{2} = \frac{2}{2}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} \quad \left(A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \right)$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

invertible $\Leftrightarrow$ $\det A \neq 0$

Adjugate Matrix for A^{-1}

Recall cofactors

$C_{ij} = \pm$ det of matrix w/ i-th row j-th column removed

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{ij} \end{bmatrix}^{T}}_{\text{"Adjugate"}}$$

Ex $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

$$|A| = +(1)\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + (2)\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -3 + 2 = -1$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{pmatrix}$$

$C_{11} = +\begin{vmatrix} -1 & 0 \\ -2 & 2 \end{vmatrix} = -2$ $\quad C_{12} = -4 = -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$ $\quad C_{13} = -3$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$ $\quad C_{22} = 1$ $\quad C_{23} = 1$

$C_{31} = 1$ $\quad C_{32} = 2$ $\quad C_{33} = 1$

$$A^{-1} = \frac{1}{(-1)} \begin{bmatrix} -2 & 0 & 1 \\ -4 & 1 & 2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$

Ex) When does the system ($s \in \mathbb{R}$)

$$3s x_1 - 5x_2 = 3$$
$$9x_1 + 5s x_2 = 2$$

have a unique solution? What is it in this case?

$$\begin{vmatrix} 3s & -5 \\ 9 & 5s \end{vmatrix} = 15s^2 + 45$$

$$= 15(s^2 + 3)$$

never zero, so always has unique solution

$$X_1 = \frac{\begin{vmatrix} 3 & -5 \\ 2 & 5s \end{vmatrix}}{15(s^2+3)} = \frac{15s+10}{15(s^2+3)}$$

$$X_2 = \frac{\begin{vmatrix} 3s & 9 \\ 9 & 2 \end{vmatrix}}{15(s^2+3)} = \frac{6s-27}{15(s^2+3)}$$

$$\vec{X} = \begin{bmatrix} \dfrac{15s+10}{15(s^2+3)} \\[2ex] \dfrac{6s-27}{15(s^2+3)} \end{bmatrix} = \frac{1}{15(s^2+3)} \begin{bmatrix} 15s+10 \\ 6s-27 \end{bmatrix}$$

Determinants $\rightsquigarrow$ signed volume

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$