MATH 212 FINAL EXAMINATION 7 MAY 2012

PART I *(4 pts each)*

(A) Show that each of the following assertions is false. Be precise, giving a specific counter-example when appropriate.

1. Let A and B be $n\times n$ symmetric matrices. Then AB is symmetric.

2. Let A be an $n\times n$ matrix. If α and β are distinct eigenvalues of A with corresponding eigenvectors x and y, respectively, then $x + y$ is an eigenvector of A.

3. If A and B are square matrices with the same characteristic polynomial, then A and B are similar.

4. Let A and B be $n\times n$ matrices. If A and B are each diagonalizable, then $A+B$ is diagonalizable.

5. There exists a 5×5 matrix, A, such that the row space of A equals the nullspace of A.

6. Let A and B be $n \times n$ matrices. If A and B are each diagonalizable, then AB is diagonalizable.

7. Let A be an $n \times n$ matrix and let B be an $n \times 1$ matrix. Then the solution space of $AX = B$, where X is an $n \times 1$ matrix, is a subspace of $\mathbf{R}^n$.

8. Let U and W be subspaces of a vector space V. Then $U \cup W$ is a subspace of V.

9. Let A be an $m \times n$ matrix, and let R be the row reduced echelon form of A. Then the column space of A equals the column space of R.

10. Let M and N be $n \times n$ nilpotent matrices. Then MN is nilpotent.

11. Let V, W, X be vector spaces and let $T: V \to W$ and $S: W \to X$ be linear transformations. Then, if ST is injective, it follows that S is injective.

12. Let *V*, *W*, *X* be vector spaces and let $T\colon V \to W$ and $S\colon W \to X$ be linear transformations. Then if *ST* is surjective it follows that *T* is surjective.

13. Let *A* be an n×n matrix. If *A* has *n* eigenvectors then *A* is diagonalizable.

14. If *A* and *B* are n×n matrices, then det(A + B) = det A + det B.

(B) Short Answer (4 pts each) No explanation is required.

1. Let A be a 19×33 matrix.
   (a) Then the *maximum* possible value of the rank of A is _____.
   (b) The *minimum* possible value of the rank of A is _____.
   (c) The *maximum* possible value of the dimension of the nullspace of A is ______.
   (d) The *minimum* possible value of the dimension of the nullspace of A is ______.

2. The dimension of the vector space of all 5×5 real symmetric matrices is _____.

3. The dimension of the vector space of all 4×4 real diagonal matrices is _____.

4. The dimension of the vector space of all real polynomials of the form $a + bx^3 + cx^5$ is _______.

5. The dimension of the vector space of all 4×3 real matrices having the property that the sum of the three entries in each of the four rows equals 0 is ______.

6. Let U and W be subspaces of a finite dimensional vector space V. Assume that dim $U = 9$ and dim $W = 5$. If $U \cap W = \{\mathbf{0}_V\}$, then the minimum possible dimension of V is ______.

7. If the characteristic polynomial of A is $p(\lambda) = \lambda^4(\lambda - 5)(\lambda - 7)$, then A is diagonalizable only if the dimension of the eigenspace associated with the eigenvalue $\lambda = 0$ is ______.

PART II *(20 pts each)*

Solve any 4 of the following 5 problems. You may answer all five for extra credit.

1. Let V be the vector space of all 3×3 real matrices.

   Define the following relation on V:

   for $A, B \in V$, *A is related to B* if the sum of the nine entries of A equals the sum of the nine entries of B.

   (a) Is this relation *reflexive*? Why?

   (b) Is this relation *symmetric*? Why?

   (c) Is this relation *transitive*? Why?

2. Let V be the vector space of all 3×3 real matrices. Let $P \in V$ be invertible. Let S be the set of all $A \in V$ such that $P^{-1}AP$ is diagonal. Is S a *subspace* of V ? If so, give a proof; if not, give a counter-example.

3. Let V and W be vector spaces and let $T: V \rightarrow W$ be a linear transformation.

   (a) Suppose that T is injective. Let $\{x_1, x_2, x_3\}$ be a linearly independent subset of V.

   Prove that $\{T(x_1), T(x_2), T(x_3)\}$ is a linearly independent subset of W.

   (b) What happens if we remove the assumption that T is injective? Explain.

4. Let V be a vector space and let $\{x, y, z\}$ be a linearly independent subset of V. Suppose that $\exists w \in V$ such that $w \notin \text{span}\{x, y, z\}$. Prove that $\{x, y, z, w\}$ is *linearly independent.*

5. Let A be an $n \times n$ matrix.

(a) Prove that A and A^T have the same characteristic polynomial.

Hint: Use the fact that, for any square matrix M, $\det M = \det M^T$.

(b) Suppose that A is invertible and that λ is an eigenvalue of A.

(i) Explain why $\lambda \neq 0$.

(ii) Prove that $1/\lambda$ is an eigenvalue of A^{-1}.

PART III *(20 pts each):* Solve any 4 of the following 5 problems. You may answer all five for extra credit.

1. *Let* $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 3 & 6 & 6 & 3 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

(a) Find the *rank* of *A.*

(b) Find a *basis for the null-space* of *A.*

(c) Find a *basis for the row space* of *A.*

(d) Find a *basis for the column* space of *A*.

2. Let T: $\mathbf{R}^2 \rightarrow \mathbf{R}^3$ be the linear transformation defined by:

$T(x, y) = (2x - 5y,\ 0,\ 10y - 4x)$

(a) Find a basis for ker(T).

(b) Find a basis for im(T).

3. Consider the following LU factorization of a matrix *A*. Using this factorization, solve the equation $AX = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

(To earn credit for this exercise, *you must use the given LU factorization*.) Show each step.

$$A = \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 5 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU$$

4. Let A be a 3×3 matrix. Suppose that (1, 2, 4) is an eigenvector of A with associated eigenvalue 5 and that (3, 1, 1) is an eigenvector of A with associated eigenvalue 7. Let v = (6, 7, 13).

(a) Find Av.

(b) Does there exist a vector $w \in \mathbf{R}^3$ such Aw = v? Either find such a vector, w, or prove that no such w can exist.

5. *Let* $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

(a) Find the *characteristic polynomial* of *A*.

(b) Is *A diagonalizable?* If so, find an invertible matrix *P* such that $P^{-1}AP$ is diagonal. If not, explain.

EXTRA EXTRA CREDIT

1. Let A and B be n×n matrices. Must AB and BA have the same eigenvalues? Give proof or counterexample.

2. If A is an n×n matrix satisfying $A^2 = -I$, what are the eigenvalues of A? If A is real, prove that n is even, and give an example.