Linear Algebra Practice Exam #1

Loyola University Chicago – Math 212.001 – Fall 2014

There will be between approx. 5 problems on your exam. In terms of points, they will be 70% computational, 20% conceptual, and 10% proof. The proofs might be low- to medium-difficulty. E.g., requiring you only to find a counter example to a statement, or do some simple linear combinations to see the answer. (See your latest workshop for examples, and #8 below for another.)

1. Solve by row-reducing an augmented matrix, displaying your solution set in multiple forms (as all coordinates (w, x, y, z) of some form, and as a particular sum of vectors):

$$\text{(a)}\quad \begin{array}{rcrcrcrcr} w & + & 2x & + & 3y & + & z & = & 1 \\ 5w & - & 6x & - & 9y & + & 5z & = & 6 \\ 6w & - & 4x & - & 6y & + & 6z & = & 7 \end{array} \qquad \text{(b)}\quad \begin{array}{rcrcrcr} a & + & b & + & c & = & 4 \\ 3a & - & b & + & 2c & = & 1 \\ 3a & - & 5b & + & c & = & -2 \end{array}$$

2. For what values of a does the system of equation with the following augmented matrix have 0, 1 or infinitely many solutions?

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 2 \\ 0 & a^2-25 & 2 & 1 \\ 1 & a^2-22 & 17 & a+3 \end{array}\right).$$

3. Consider the differential equation(s):

$$f'(x) = 2f(x) + 2x^n - 1. \qquad (\star_n)$$

   (a) Find all polynomials $f(x)$ in $\mathcal{P}_2$ satisfying $(\star_2)$.

   (b) Find all polynomials $f(x)$ in $\mathcal{P}_2$ satisfying $(\star_3)$.

4. Indicate if the following statements are True or False:

   (a) It is never the case that $X = [X]$.

   (b) It is always the case that $[[X]] = [X]$.

   (c) If X is linearly independent, then X is a basis for $[X]$.

   (d) If $X \subseteq V$ for some vector space V, and X is linearly independent, then X is a basis for V.

   (e) If $X \subseteq V$ for some vector space V, and $[X] = V$, then X is a basis for V.

   (f) If $X \subseteq V$ for some vector space V of dimension n, and $|X| = n$, and X is linearly independent, then X is a basis for V.

   (g) If $X \subseteq V$ for some vector space V of dimension n, and $|X| = n$, and $[X] = V$, then X is a basis for V.

(h) A system of equations with more equations than unknowns never has a solution.

(i) A system of equations with more unknowns than equations will never have exactly one solution.

(j) If A is an $n \times n$ matrix and the matrix equation $A\vec{x} = \vec{b}$ has a non-trivial solution, then the Gauss-Jordan form of A has n pivots.

(k) If A is an invertible $n \times n$ matrix and $\vec{b}$ is a vector in $\mathbb{R}^n$, then the matrix equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

(l) (5) True or False: If V is a vector space, every subspace of V must contain the vector $\vec{0}$.

(m) (5) True or False: The set of $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : a = b + 1 \right\}$ is not a subspace of $\mathbb{R}^3$.

(n) It is possible to find 5 linearly independent vectors in $\mathbb{R}^3$.

(o) (Definition: The *null space* of a matrix is the set of vectors $\vec{x}$ solving the homogeneous system $A\vec{x} = \vec{0}$.) If A is a singular $n \times n$ matrix, then the null space of A is just $\{\vec{0}\}$.

(p) (Definition from class: a matrix is *row/column deficient* if it has fewer pivots than rows/columns in its Gauss-Jordan form.) If A is a row-deficient matrix, then the null space of A is never just $\{\vec{0}\}$.

(q) If A is a column-deficient matrix, then the null space of A is never just $\{\vec{0}\}$.

(r) The set of polynomials such that $f(1) = f(2)^2$ is a subspace of the vector space $\mathcal{P}$ of all polynomials.

5. Show that the set of 2×2 matrices A such that the trace satisfies $\text{tr}(A) = 0$ is a subspace of $M_{2\times 2}$.

6. Find a subset X' of the vectors X that forms a basis for $[X]$. If $[X] \neq \mathcal{P}_3$, then extend X' to a basis for $\mathcal{P}_3$.

$$X = \left\{1 + x + x^2 + x^3, 1 - x + x^2 - x^3, 1 + x^2, -x - x^3\right\}$$

7. Find a tidy set Y of vectors in $\mathbb{R}^4$ that forms a basis for $[X]$. If $[X] \neq \mathbb{R}^4$, then extend Y to a basis for $\mathbb{R}^4$.

$$X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

8. (Definition: a *polynomial* $\vec{p}$ is a finite collection of monomials, e.g., $\vec{p} = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r$. The highest power of x occurring in $\vec{p}$ with nonzero coefficient is called the *degree* of $\vec{p}$.) Show that the vector space $\mathcal{P}$ of all polynomials is infinite dimensional. *(Hint: argue by contradiction, starting with a finite basis for $\mathcal{P}$.)*