Final Exam Review Problems Math 212 (Fall 2014)

In addition to your looking over your midterms, consider working through these problems as you study for your final exam: **Monday 12/15!**

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1. At noon ($t = 0\,\text{min}$), the minute and hour hands of a clock coincide. What is the next time when:

   (a) they are perpendicular? (b) they coincide?

2. Is this a vector space?: The set of functions, $f(x)$, for which there is at least one solution $u(x)$ to the differential equation $u'' - xu = f(x)$.
   [Note: You are not being asked to solve this differential equation.]

3. This is not a vector space, for most f: The set of solutions, $u(x)$, to the differential equation $u'' - xu = f(x)$. (Can you see why?) Put a condition on f that makes this set a vector space.
   [Note: You are not being asked to solve this differential equation.]

4. Find a basis for this subspace of $\mathcal{M}_{2\times 3}(\mathbb{R})$, those $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose kernel contains $\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$.

5. Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W = U + V := \{u + v \mid u \in U \text{ and } v \in V\}$.

   (a) Show that W is a linear space.

   (b) Find all possible values for $\dim W$.

6. Let $A\colon \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

   (a) A is injective (hence $n \leq k$).

   (b) $\text{nullity}(A) = 0$.

   (c) A has a left inverse B, so $BA = I$.

   (d) The columns of A are linearly independent.

7. Let $A\colon \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

   (a) A is surjective (hence $n \geq k$).

   (b) $\text{rank}(A) = k$.

   (c) A has a right inverse B, so $AB = I$.

   (d) The columns of A span $\mathbb{R}^k$.

8. Let A and B be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

   (a) Either $A = 0$ or $B = 0$ (or both).

   (b) $BA = 0$.

   (c) If $\det A = -3$, then $B = 0$.

   (d) If B is invertible then $A = 0$.

   (e) There is a vector $\vec{v} \neq 0$ such that $BA\vec{v} = 0$.

9. For what value(s) of C does the linear system below have:

   (a) a unique solution? (b) no solution? (c) infinitely many solutions?

   (Justify your assertions.)

$$\left\{\begin{array}{rcrcrcl} Cx & + & y & + & z & = & 1 \\ x & + & Cy & + & z & = & 1 \\ x & + & y & + & Cz & = & 1 \end{array}\right.$$

10. Let $X = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$. Find:

    (a) a subset of X that is a basis for $[X]$,

    (b) a tidy basis for $[X]$,

    (c) a basis for the vector space of tuples $\left\{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid \sum_i a_i \vec{x}_i = 0\right\}$. Also, prove that this set is indeed a vector space.

11. Give a proof or counterexample the following. In each case your answers should be brief.

    (a) Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors in a vector space V and $t\colon V \to V$ is a linear map.

    - If $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly dependent, then $t(\vec{u})$, $t(\vec{v})$ and $t(\vec{w})$ are linearly dependent.
    - If $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent, then $t(\vec{u})$, $t(\vec{v})$ and $t(\vec{w})$ are linearly independent.
    - If $t(\vec{u})$, $t(\vec{v})$ and $t(\vec{w})$ are linearly independent, then $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent.

    (b) If $t\colon \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map, it is possible that the nullspace of t is one dimensional.

12. Suppose A, B are two real matrices.

    (a) If A, B are $n \times n$ and satisfy AB is invertible, show that A and B are also invertible.

    (b) If A is 3×2 and B is 2×3, show that AB cannot be invertible.

    (c) If A is 3×2 and B is 2×3, show that BA could be invertible.

13. Let L, M, and N be linear transformations on $\mathbb{R}^2$, given in terms of the standard basis by:

    - $L\vec{e}_1 = \vec{e}_2, \qquad L\vec{e}_2 = -\vec{e}_1$ (rotation by 90 degrees counterclockwise)
    - $M\vec{e}_1 = -\vec{e}_1, \quad M\vec{e}_2 = \vec{e}_2$ (reflection across the vertical axis)
    - $N\vec{e}_1 = -\vec{e}_1, \quad N\vec{e}_2 = -\vec{e}_2$ (reflection across the origin)

    (a) Draw pictures describing the actions of the composite maps LM, LN, ML, MN, NL, and NM.

    (b) Which pairs of maps commute?

    (c) Which of the following identities are correct—and why?

    - $L^2 = N$
    - $N^2 = I$
    - $L^4 = I$
    - $L^5 = L$
    - $M^2 = I$
    - $M^3 = M$
    - $MNM = N$
    - $NMN = L$

    (d) Find matrices representing each of the linear maps L, M, N.
    [Note: You shouldn't/needn't do this before answering the other parts... but you could.]

14. Let $S \subseteq \mathbb{R}^3$ be the subspace spanned by the two vectors $\vec{v}_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$ and let T be the orthogonal complement of S.

    (a) Find an orthogonal basis $\{\vec{w}_1, \vec{w}_2\}$ for S and use it to find the 3×3 matrix A (in the standard basis) that projects vectors orthogonally into S.

    (b) Find an orthogonal basis $\{\vec{w}_3\}$ for T and use it to find the 3×3 matrix B that projects vectors orthogonally into T.

    (c) Verify that $A = I - B$. How could you have seen this in advance?

    (d) Sometimes, a different basis is best. Let $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$. Show that

    $$\operatorname{Rep}_{\mathcal{B},\mathcal{B}} A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \end{pmatrix} \quad \text{and} \quad \operatorname{Rep}_{\mathcal{B},\mathcal{B}} B = \begin{pmatrix} & & \\ & & \\ & & 1 \end{pmatrix},$$

    making the previous part especially easy to see.

(e) Put $P = \left[\, \vec{w}_1 \,|\, \vec{w}_2 \,|\, \vec{w}_3 \,\right]$. What does $P^{-1}AP$ look like?
[Note: You shouldn't have to do any matrix computations, at all, to answer this question.]

15. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

(a) What is the dimension of the image of A? Why?

(b) What is the dimension of the kernel of A? Why?

(c) What are the eigenvalues of A? Try to do this by inspection, but also compute the characteristic polynomial, for practice, to verify your answer.

(d) What are the eigenvalues of $B = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$?

[Hint: $B = A + 3I$].

16. Let $t\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the function given by

$$t(v, w, x, y, z) = (v - x + z, w - x + y - z, v + w + y - 2x).$$

Find bases $\mathcal{B}$ and $\mathcal{C}$ so that $\mathrm{Rep}_{\mathcal{B},\mathcal{C}}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

17. Compute the determinant by reducing to triangular form: $\begin{vmatrix} 20 & 40 & 50 & -10 \\ 1 & 3 & 4 & -2 \\ 3 & 7 & 9 & -2 \\ -3 & -7 & 9 & 4 \end{vmatrix}$.

18. Find:

(a) A 3×3 matrix whose minimal polynomial is x^2.

(b) The minimal polynomial of the transformation $t\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by:

$$p(x) \mapsto p(x+1).$$

(So, e.g., $x^3 \mapsto (x+1)^3 = 1 + 3x + 3x^2 + x^3$.)

(c) the possible minimal polynomials of a 5×5 matrix with characteristic polynomial $(x-1)^2(x-2)^3$.

(d) the possible Jordan forms (up to similarity) of a matrix with characteristic polynomial $(x-1)^2(x-2)^4$ and minimal polynomial $(x-1)^2(x-2)^4$.