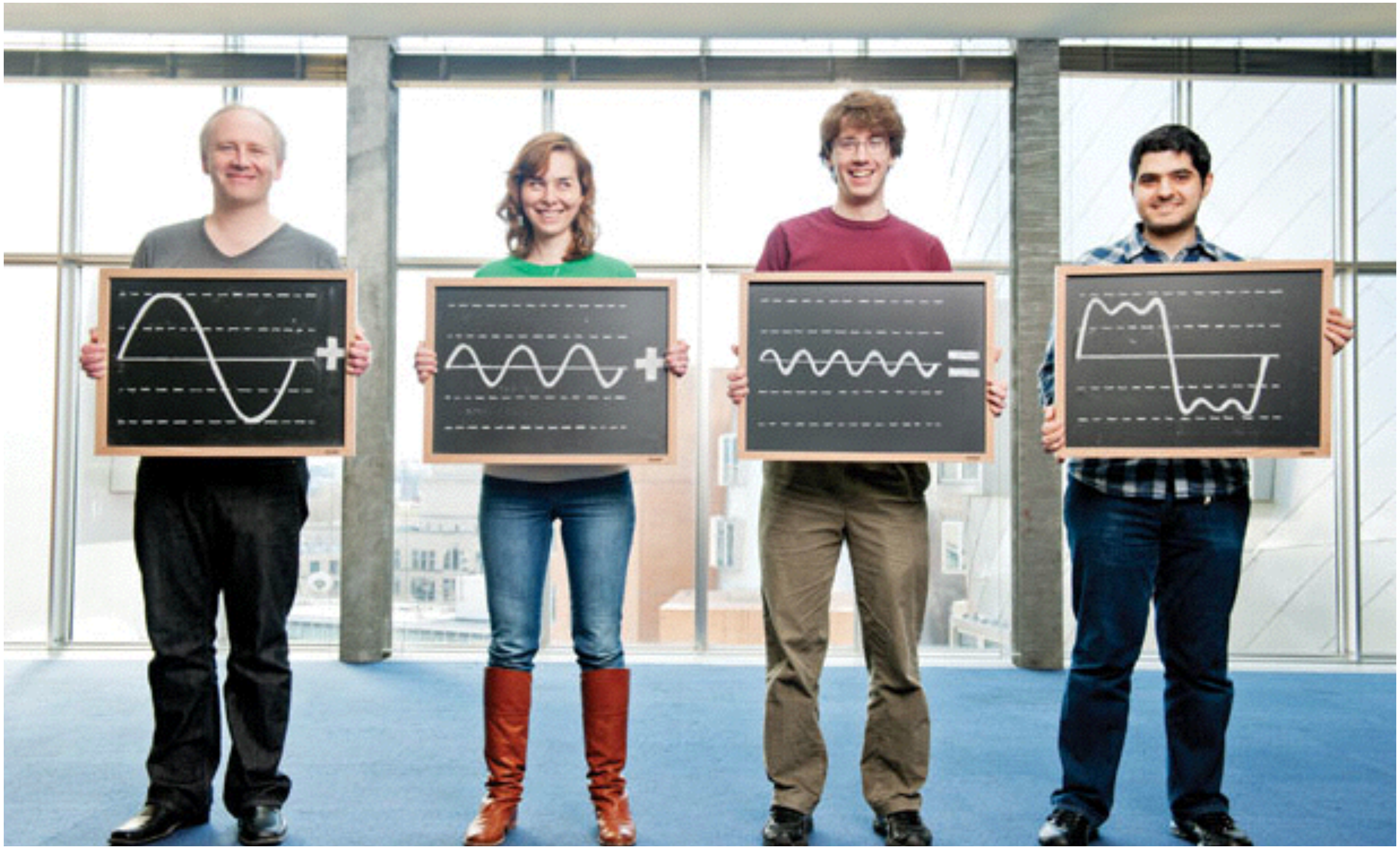
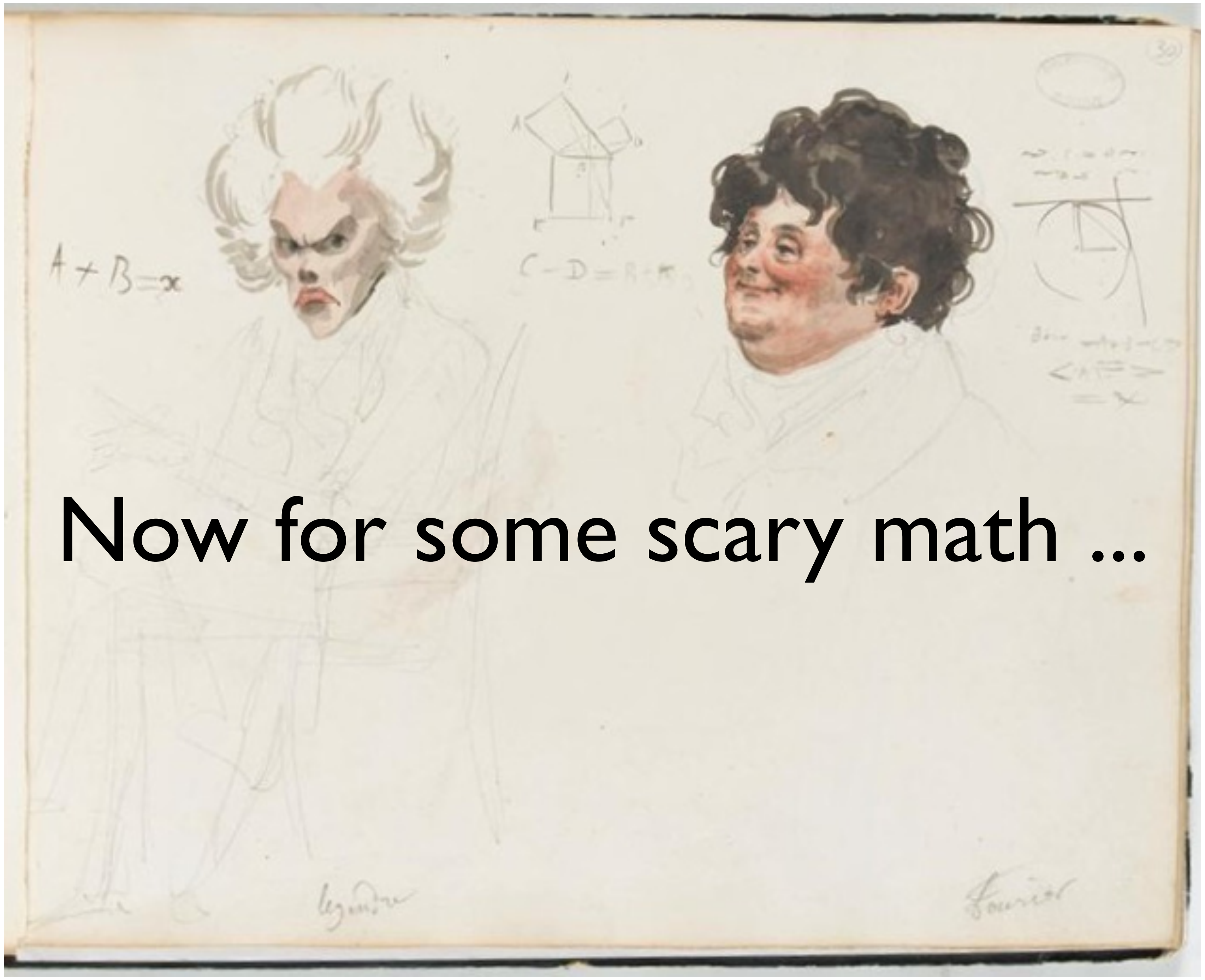


Fourier Transform



Now for some scary math ...

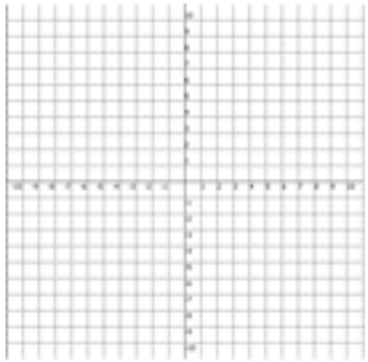
Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

what's this? *what's this?*



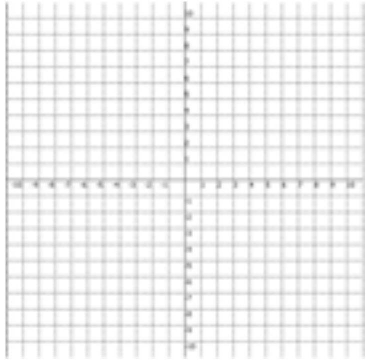
Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



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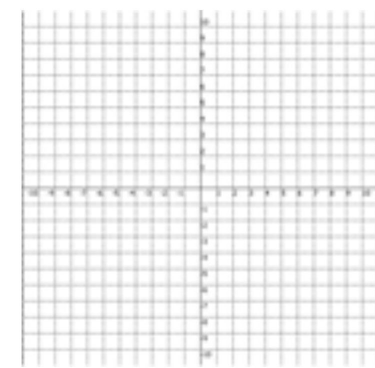
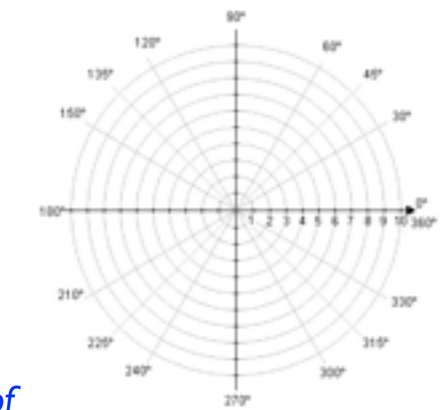




what kind of transform is this?

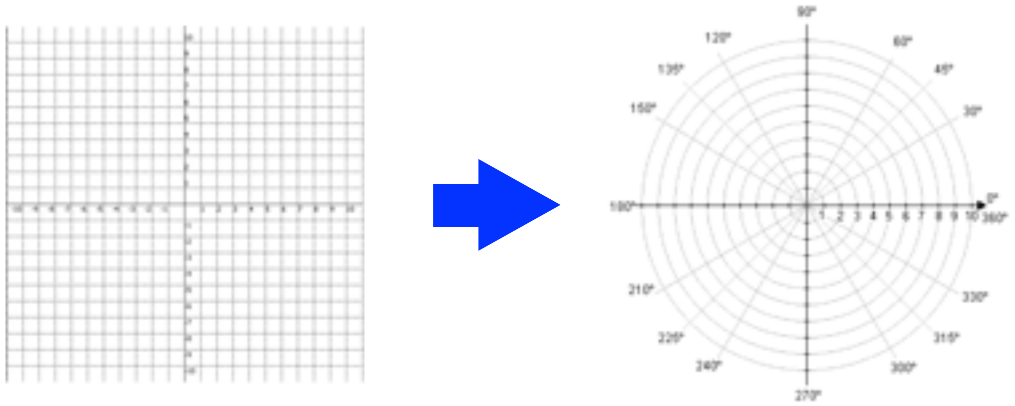
Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Polar

Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

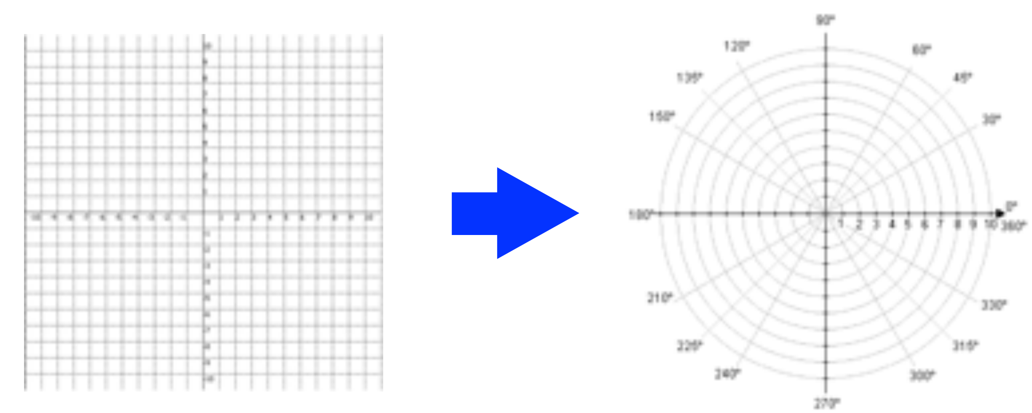
real imaginary

Alternative re-parameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

How do you compute r and theta?



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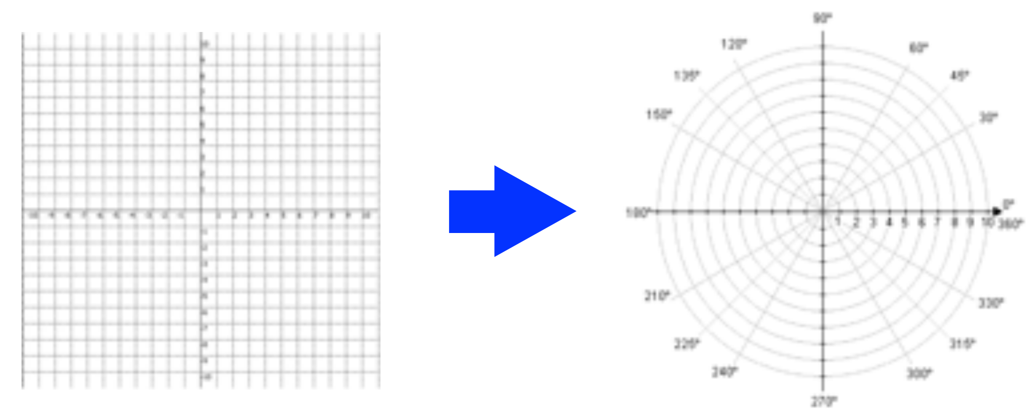

Alternative re-parameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \qquad r = \sqrt{R^2 + I^2}$$

Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

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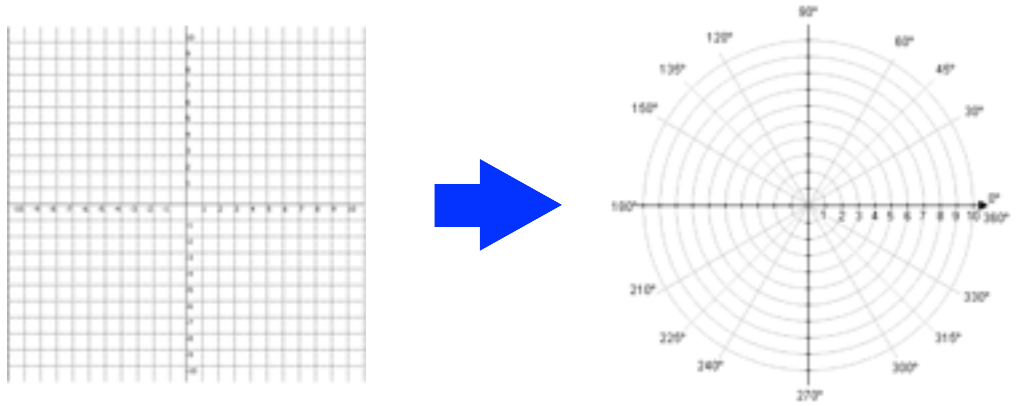


How do you write this in exponential form?

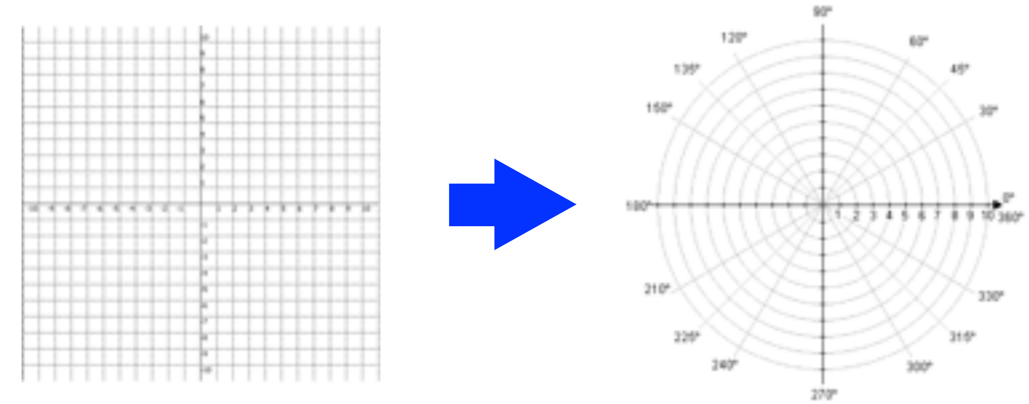
Recalling some basics...

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative re-parameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \qquad r = \sqrt{R^2 + I^2}$$

OR

exponential form

$$re^{j\theta}$$

'Euler's formula'

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This will help us understanding of the Fourier transform equations ...

Fourier transform | Inverse Fourier transform

Continuous

$$F(k) = \int_{\infty}^{-\infty} f(x) e^{-j2\pi kx} dx$$

$$f(x) = \int_{\infty}^{-\infty} F(k) e^{j2\pi kx} dk$$

Discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

$k = 0, 1, 2, \ldots, N-1$

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$x = 0, 1, 2, \ldots, N-1$

Where is the connection to the 'summation of sine waves' idea?

	Fourier transform	Inverse Fourier transform
Continuous	$F(k) = \int_{\infty}^{-\infty} f(x) e^{-j2\pi kx} dx$	$f(x) = \int_{\infty}^{-\infty} F(k) e^{j2\pi kx} dk$
Discrete	$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$ $k = 0, 1, 2, \ldots, N-1$	$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$ $x = 0, 1, 2, \ldots, N-1$

Where is the connection to the 'summation of sine waves' idea?

Where is the connection to the 'summation of sine waves' idea?

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

'Euler's formula'

$$e^{j\theta} = \cos\theta + j\sin\theta$$

scaling parameter

$$f(x) = \sum_{k=0}^{N-1} F(k) \left\{ \cos(2\pi kx) + j\sin(2\pi kx) \right\}$$

sum over frequencies

wave components

“So how do you actually compute the DFT?”

–A. Student

Computing the Discrete Fourier Transform…

$$F(k) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-j2\pi kx/N}$$

…is just a matrix multiplication.

$$\boldsymbol{F} = \boldsymbol{W}\boldsymbol{f}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}$$

$$W = e^{-j2\pi/N} \qquad W = W^{2N}$$

Example

input signal

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 0 \end{bmatrix}$$

DFT

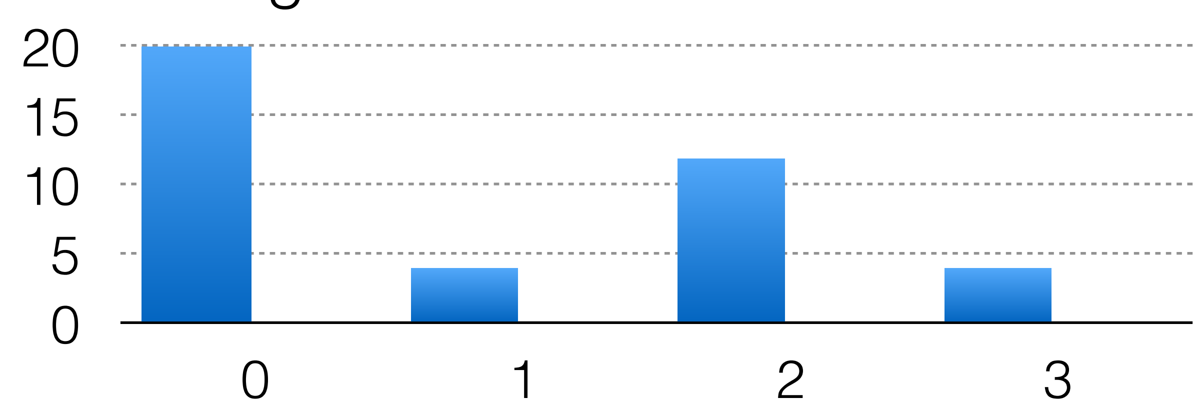
$$F(k) = \sum_{x=0}^{3} f(x) e^{-j2\pi xk/4}$$

$$= \sum_{x=0}^{3} f(x)(-j)^{xk}$$

Frequency Domain representation

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{bmatrix}$$

Magnitude of DFT coefficients



The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathcal{F}\{g \star h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} \star \mathcal{F}^{-1}\{h\}$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!