ECE 511 Analysis of Random Signals

Homework 1

IIT, Chicago

Instructor: Salim El Rouayheb
Assigned on: Wed Aug 30, 2014
Due on: Wed Sep 3, 2014
Logistics: I will collect the students' papers during the first five minutes of the lecture. You also have the option of submitting your solution electronically through Blackboard. Only scanned copies of your paper will be accepted on Blackboard, pictures taken by phone will not be graded. I will not accept any submission, by hand or on Blackboard, after 3:20 pm CDT on the due date.

Submit your solution to the following problems from Stark and Woods (fourth edition) and the extra three problems:

Pages 66-77: 1.29, 1.32, 1.45, 1.56, 1.60, 1.64, 1.66, 1.69

Problem 9: Communication through a Series of Relays
A transmitter is trying to send a receiver a single bit, either a 0 or a 1. When the bit is transmitted, it goes through a series of n relays before it is received. Each relay flips the bit independently with probability p.

1. Prove that the probability the bit is received correctly is
$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k}(1-p)^{n-2k}.$$

2. We consider an alternative way to calculate this probability. Let us say the relay has a *bias* q if the probability it flips the bit is $(1-q)/2$. The bias q is therefore a real number in the range $[-1, 1]$. Prove that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 q_2$.

3. Prove that the probability that the bit is received correctly when it passes through n relays as described before in (1) is
$$\frac{1+(1-2p)^n}{2}.$$
Compare with the result in (1).

Problem 10: What Happens in Vegas
Because of the stress of the first week of school, you decide to take a little vacation and go to Las Vegas on Labor Day weekend. There, you want to spend only $m = \$100$ on gambling. But, being an inexperienced gambler, you choose to play the following simple game. A dealer tosses a fair coin. If it

comes up heads you win a dollar, but if it comes up tails you pay the dealer one dollar. You decide to play this game repeatedly until either you win $N = \$1000$, or you loose all your money. We want to find the probability of the bad event B of you ultimately losing all your money.

1. Let $p_m = P_m(B)$ be the probability of losing all your money when you start the game with m dollars. Show that $p_m = \frac{1}{2}(p_{m+1} + p_{m-1})$. (Hint: Condition on the outcome of the first coin toss.)
2. Find the expression of $P_m(B)$ as a function of m and N. What is the consequence of being greedy and increasing N? (Hint: Start by checking that $p_m - p_{m-1}$ is a constant independent of m.)

Problem 11: Cliques in Random Graphs

Recall the definition in class of a random graph $G(n, p)$ on n vertices, where each edge occurs independently with probability $p = 1/2$. A *clique* in a graph is a set of vertices in which any two vertices are connected by an edge. The *clique number* of a graph G denoted, $\omega(G)$, is the maximum number of vertices in a clique in G. For example, in the graph G in Fig. 1 the following sets $\{7, 6\}, \{1, 2, 7\}$ and $\{3, 4, 5, 6\}$ are cliques, but $\{1, 2, 3, 7\}$ is not a clique. G has a clique of size 4, but no clique of size 5, so $\omega(G) = 4$. The objective of this exercise is to determine how large can the clique number of a random graph be as a function of n.

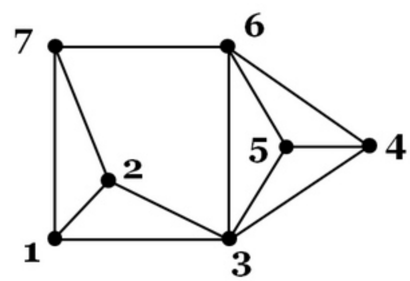

Figure 1: Graph with clique number $\omega(G) = 4$.

1. Show that the probability that the clique number of $G(n, 1/2)$ is at least k is upper bounded by $\binom{n}{k} 2^{-\binom{k}{2}}$, i.e.,
$$P\left(\omega\Big(G(n, k)\Big) \geq k\right) \leq \binom{n}{k} 2^{-\binom{k}{2}}.$$
2. Show that the probability that $G(n, 1/2)$ has a clique of size strictly larger than $2 \log_2 n$ goes to zero as the number of vertices n goes to infinity. (Hint: start by showing that $\binom{n}{k} \leq n^k$).
3. Comment on the result above.