ECE 511 Analysis of Random Signals

Homework 1

IIT, Chicago

Instructor: Salim El Rouayheb
Assigned on: Wed Sep 10, 2014
Due on: Wed Sep 22, 2014
Logistics: I will collect the students' papers during the first five minutes of the lecture. You also have the option of submitting your solution electronically through Blackboard. Only scanned copies of your paper will be accepted on Blackboard, pictures taken by phone will not be graded. I will not accept any submission, by hand or on Blackboard, after 3:20 pm CDT on the due date.

Submit your solution to the following problems from Stark and Woods (fourth edition) and the extra seven problems:

Pages 141-149: 2.7, 2.14, 2.17, 2.36, 2.37 (6 points each)

Problem 6: Memoryless distributions
(10 points) The probability distribution of a random variable X is called memoryless, if for any positive numbers m and n,

$$P(X > m + n | X > m) = P(X > n).$$

Show that the geometric and exponential distributions are memoryless.

Problem 7: A computer reserves a path...
(10 points) A computer reserves a path in a network for 10 minutes. To extend the reservation the computer must successfully send a "refresh" message before the expiry time. However, messages are lost with probability $1/2$. Suppose that it takes 10 seconds to send a refresh request and receive an acknowledgment. When should the computer start sending refresh messages in order to have a 99% change of successfully extending the reservation time?

Problem 8: Repetition coding
(10 points) A modem transmits over an error-prone channel, so it repeats every "0" or "1" bit transmission five times. We call each such group of five bits a "codeword." The channel changes an input bit to its complement with probability p = 1/10 and it does so independently of its treatment of other input bits. The modem receiver takes a majority vote of the five received bits to estimate the input signal.

1. Find the probability that the receiver makes the wrong decision.
2. The modem transmits 1000 5-bit codewords.What is the average number of codewords in error? If the modem transmits 1000 bits individually without repetition, what is the average number of bits in error? Explain how error rate is traded off against transmission speed.

Problem 9: Disk failure in a data center

(10 points) A data center has 10,000 disk drives. Suppose that a disk fails in a given day with probability 10^{-3}.

1. Find the probability that there are no failures in a given day.
2. Find the probability that there are fewer than 10 failures in two days.
3. Find the number of spare disk drives that should be available so that all failures in a day can be replaced with probability 99%.

Problem 10: Gaussian distribution

(10 points) A binary transmission system transmits a signal $X(-1$ to send a "0" bit; $+1$ to send a "1" bit). The received signal is $Y = X + N$ where noise N has a zero-mean Gaussian distribution with variance σ^2. Assume that "0" bits are three times as likely as "1" bits.

1. Find the conditional pdf of Y given the input value: $f_Y(y|X = +1)$ and $f_Y(y|X = -1)$.
2. The receiver decides a "0" was transmitted if the observed value of y satisfies

   $$f_Y(y|X = -1)P[X = -1] > f_Y(y|X = +1)P[X = +1]$$

   and it decides a "1" was transmitted otherwise. Use the results from part 1 to show that this decision rule is equivalent to: If $y < T$ decide "0"; If $y >= T$ decide "1".
3. What is the probability that the receiver makes an error given that a $+1$ was transmitted? a -1 was transmitted? Assume $\sigma^2 = 1/16$.
4. What is the overall probability of error?

Problem 11: On-off Keying in Optical Communications

(10 points) The receiver in an optical communications system uses a photodetector that counts the number of photons that arrive during one time unit. Suppose that the number X of photons can be modeled as a Poisson random variable with rate λ_1 when a signal is present (say bit 1 is transmitted) and a Poisson random variable with rate $\lambda_0 < \lambda_1$ when a signal is absent (say bit 0 is transmitted). Let p denote the probability that the transmitted bit is 1.

1. What is the probability that a bit 1 was transmitted if the receiver detects k photons?
2. To decide the value of the transmitted bit, the receiver implements the following decision rule, called Maximum Likelihood (ML) detection: if $P[\text{bit} = 1|X = k] > P[\text{bit} = 0|X = k]$, decide bit 1 was sent, otherwise bit 0. Find the corresponding threshold decision rule based on the value of X.
3. What is the probability of error for the above decision rule.

Problem 12: Movie download [from Exam 1, Fall 2013]

(10 points) A computer wants to download a movie consisting of n packets, $P_1, \ldots, P_n$, hosted on a server. The server works in a peculiar way. Every time slot after the computer has initiated the download request, the server sends the computer a packet chosen independently and uniformly at random from the

set of n packets. We wish to find the average amount of time that will take the computer to download the entire movie.

Let $T_i, i = 1, \ldots, n$, be a random variable that denotes the time (in slots) to download a new packet, not already downloaded, after $i-1$ distinct packets have been downloaded. For example, if the computer has successively downloaded packets $\mathbf{P_4}, P_4, P_4, \mathbf{P_2}, P_4, P_2, P_2, \mathbf{P_9}, \mathbf{P_1} \ldots$, so far, then $T_1 = 1, T_2 = 3, T_3 = 4, T_4 = 1$.

1. Find the pmf of T_i.
2. Let T the random variable that denotes the time to download the entire movie. Express T as a function of $T_1, T_2, \ldots, T_n$.
3. Using the previous two results, show that $E(T) \approx n \log n$. (You may want to use the following approximation of the harmonic number $H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \log n$.)
4. Comment on this result. How good/bad is the average download time as a function of n?