ECE 511 Analysis of Random Signals

Homework 4

IIT, Chicago

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Assigned on: Wed Oct 22, 2014
Due on: Mon Oct 29, 2014
Logistics: I will collect the students' papers during the first five minutes of the lecture. You also have the option of submitting your solution electronically through Blackboard. Only scanned copies of your paper will be accepted on Blackboard, pictures taken by phone will not be graded. I will not accept any submission, by hand or on Blackboard, after 3:20 pm CDT on the due date.

Submit your solution to the following problems from Stark and Woods (fourth edition) AND the extra problems. .
Pages 284-293: 4.38, 4.39, 4.43, 4.50, 4.51, 4.54, 4.56, 4.57, 4.58, 4.61, 4.74

Problem 12: MMSE

Let X and Y be two r.v. with the following joint pdf

$$f_{X,Y}(x,y) = k(x+y), \text{ for } 0 \leq x \leq 1,\ 0 \leq y \leq 1.$$

1. Find k.
2. Find the Linear Minimum Mean Square Error (LMMSE) estimator of Y given X.
3. Find the Minimum Mean Square Error (MMSE) estimator of Y given X.
4. Compare the mean square error of the estimators in parts 1 and 2.

Problem 13: A Facebook's Sever

The number of tasks N that a server in a Facebook data center receives in an hour is a geometric random variable with parameter p. T he execution time of task i, denoted by $X_i,\ i = 1, 2, \ldots, N$, is exponentially distributed r.v. with mean $\frac{1}{\alpha}$. We assume that the X_i's are i.i.d. We are interested in $S = \sum_{i=1}^{N} x_i$, the total time that the server will take to finish all the tasks submitted in an hour.

1. Find $E[S]$.
2. Let $\Phi_S(\omega)$ be the characteristic function of S. Prove that

$$\Phi_S(\omega) = M_N(\Phi_{X_1}(\omega)).$$

3. Find the pdf of S by using the result in 2.

Problem 14: When Chernoff met Bernoulli

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $P(X_i = 1) = p_i, i = 1, \ldots, n$. Let $X := X_1 + \cdots + X_n$ and $\mu := E(X)$. We want to prove the following Chernoff bound on the tail distribution of X:

$$P\left(X \geq (1+\delta)\mu\right) \leq e^{-\mu\delta^2/3}, \tag{1}$$

for any $\delta \in (0, 1]$.

1. Show that the moment generating function of each X_i satisfies $M_{X_i}(t) \leq e^{p_i(e^t-1)}$. (You may want to use the fact that for any $y, 1 + y \leq e^y$).
2. Show that $M_X(t) \leq e^{(e^t-1)\mu}$.
3. Using the Chernoff bound of Eq. (4.6-4) in the textbook, show that
$$P(X \geq (1+\delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu. \tag{2}$$
4. Check that $\frac{e^\delta}{(1+\delta)^{(1+\delta)}} \leq e^{-\delta^2/3}$. Deduce Eq. (1).
5. Write a Matlab code that compares the Chernoff bound in Eq. (1) to the Chebyshev bound, the Markov bound, CLT and the exact expression for $n = 5, n = 10, n = 20$ and $n = 50$ for $\delta \in (0, 1]$. You should submit your Matlab code and 4 matlab figures corresponding to the 4 values of n. Each figure should have the 5 required curves.