ECE 511 Analysis of Random Signals

Homework 5

IIT, Chicago

Instructor: Salim El Rouayheb
Assigned on: Mon Nov 3, 2014
Due on: Mon Nov 10, 2014
Logistics: There is no class on Monday. You can only submit your solutions through blackboard. The deadline is Monday midnight. I will only accept scanned papers. No phone pictures.

Problem 1

Let U_1, U_2 and U_3 be independent zero-mean, unit-variance Gaussian random variables and let $X = U_1$, $Y = U_1 + U_2$, and $Z = U_1 + U_2 + U_3$.

1. Find the covariance matrix of (X, Y, Z).
2. Find the joint pdf of (X, Y, Z).
3. Find the conditional pdf of Y and Z given X.
4. Find the conditional pdf of Z given X and Y.

Problem 2

A three-dimensional vector random variable, X, has a covariance matrix of

$$K = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 5 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Find a transformation matrix A such that the new random variables $Y = AX$ will be uncorrelated.

Problem 3

Suppose X, Y, and Z are jointly Gaussian random variables with mean vector

$$E\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix},$$

and covariance matrix

$$K = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Find P($X > 2Y - 3Z$).

Problem 4

Let X_1, X_2 and X_3 be a set of zero-mean Gaussian random variables with a covariance matrix of the form

$$K = \sigma^2 \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}.$$

1. Find $E[X_1|X_2 = x_2, X_3 = x_3]$.
2. Find $E[X_1 X_2|X_3 = x_3]$.
3. Find $E[X_1 X_2 X_3]$.

Problem 5

Let X, Y, and Z have joint pdf

$$f_{X,Y,Z}(x, y, z) = k(x + y + z) \text{ for } 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1.$$

1. Find the minimum mean square error linear estimator for Y given X and Z.
2. Find the minimum mean square error estimator for Y given X and Z.
3. Compare the mean square error of the estimators in parts 1 and 2.

Problem 6

A receiver in a multiuser communication system accepts k binary signals from k independent transmitters: $\mathbf{Y}$=$(Y_1, Y_2, \ldots, Y_k)$, where Y_k is the received signal from the Kth transmitter. In an ideal system the received vector is given by

$$\mathbf{Y} = \mathbf{A}\mathbf{b} + \mathbf{N},$$

where $\mathbf{A} = [\alpha_k]$ is a diagonal matrix of positive channels gains $\alpha_1, \ldots, \alpha_k$ forming the diagonal, $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is the vector of bits from each of the transmitters where $b_i = \pm 1$, and $\mathbf{N}$ is a vector of k independent zero-mean unit-variance Gaussian random variables. Assume the transmitted bits b_k are independent and equally likely +1 or -1.

Find the minimum mean square linear estimator for $\mathbf{B}$ given the observation $\mathbf{Y}$. How can this estimator be used in deciding what where the transmitted bits?

Problem 7:

Let ω be a random variable uniformly distributed on the interval $(0, 1]$. Determine in which of the four senses (a.s., p., m.s, d) if any, each of the following two sequences of random variables converges. Justify your answers.

1. $X_n(\omega) = \frac{(-1)^n}{n\sqrt{\omega}}$.
2. $X_n(\omega) = n(\omega)^n$.

Problem 8:

Let θ be uniformly distributed on the interval $[0, 2\pi]$. In which of the four senses (a.s., p., m.s, d) do each of the following two sequences converge. Identify the limits, if they exist, and justify your answers.

1. $(X_n : n > 1)$ defined by $X_n = \cos(n\Theta)$.
2. $(Y_n : n > 1)$ defined by $Y_n = |1 - \frac{\Theta}{\pi}|^n$.

Problem 9:

Let $U_1, U_2, \ldots$ be a sequence of independent random variables, with each variable being uniformly distributed over the interval $[0, 2]$, and let $X_n = U_1 U_2 \cdots U_n$ for $n \geq 1$.

Determine in which of the senses (a.s., p., m.s, d) the sequence (X_n) converges as $n \to \infty$, and identify the limit, if any. Justify your answer. (Hint: Use the LLN).

Problem 10:

Let $B_1, B_2, \ldots$ be a sequence of independent Bernoulli(p) random variables for some parameter p with $0 < p < 1$. Let $Y_0 = 1$ and for each $n \geq 1$ let

$$Y_n = \begin{cases} 2Y_n - 1 & \text{if } B_n = 0 \\ 1 & \text{if } B_n = 1. \end{cases}$$

The variables $(Y_k : k \geq 0)$ in this problem might model the payout of a daily lottery that doubles from one day to the next whenever there is no winner.

1. Find the pmf of Y_n for general $n \geq 0$. (Hint: Begin by thinking about the possible values of Y_n, and what values of the B variables give rise to them.)
2. Does Y_n converge in distribution? Justify your answer, and if it does, identify the limit.
3. Does Y_n converge in probability? Justify your answer, and if it does, identify the limit.