UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 3
Fall 2014

Issued: Friday, September 19 **Due:** Thursday, September 25 (beginning of class)

Problem 3.1
Let $X_1, \ldots, X_n$ be i.i.d. absolutely continuous variables with common density f_θ, $\theta \in \mathbb{R}$, given by

$$f_\theta(x) = \begin{cases} \frac{\phi(x)}{\Phi(\theta)}, & x < \theta; \\ 0, & x \geq \theta. \end{cases}$$

(This is the density for the standard normal distribution truncated above at θ.)

(a) Derive a formula for the UMVU for $g(\theta)$. (Assume that g is differentiable and behaves reasonably as $\theta \to \pm\infty$.)

(b) If $n = 3$ and the observed data are $-2.3, -1.2$, and 0, what is the estimate for θ^2?

Problem 3.2
Consider a scale family $\frac{1}{\theta} f(x/\theta)$, $\theta > 0$ where f is some fixed density function.

(a) Show that the amount of information that a single observation X contains about θ is given by

$$\frac{1}{\theta^2} \int \left[\frac{y f'(y)}{f(y)} + 1 \right]^2 f(y) dy.$$

(b) Show that the information X contains about $\xi = \log \theta$ is independent of θ.

(c) For the Cauchy distribution $C(0, \theta)$, show that $I(\theta) = 1/(2\theta^2)$.

Problem 3.3
(Poisson birth process) This example illustrates some differences that can arise with dependent data, as opposed to i.i.d. sampling models. Consider a random sequence $Y_0, Y_1, \ldots, Y_n$ such that $Y_0 \sim \text{Poi}(\theta)$, and Y_j given the past $(Y_0, \ldots, Y_{j-1})$ is also Poisson with mean θY_{j-1}. The maximum likelihood estimate (MLE) of θ maximizes the log likelihood $\ell(\theta) = \log p(Y_0, \ldots, Y_n; \theta)$ of the data.

(a) Show that the MLE of θ based on $(Y_0, \ldots, Y_n)$ is given by $\widehat{\theta} = (\sum_{j=0}^{n} Y_j)/(1 + \sum_{j=0}^{n-1} Y_j)$.

(b) Show that the information in $(Y_0, \ldots, Y_n)$ about θ is given by $I(\theta) = \theta^{-2}(\theta + \theta^2 + \ldots + \theta^{n+1})$. What happens to this information for $\theta < 1$? Intuitively, what is happening in this model?

Problem 3.4
Suppose that the vector $X = (X_1, \ldots, X_n)$ has i.i.d. elements with the density

$$p(x;\theta) \;=\; \exp(\theta - x), \qquad \text{for } x \geq \theta.$$

Let $\delta(\cdot)$ be any unbiased estimator of θ based on X.

(a) Using Cauchy-Schwartz, first show that

$$\operatorname{var}_\theta(\delta(X)) \;\geq\; \sup_{\theta' \geq \theta} \frac{(\theta - \theta')^2}{\mathbb{E}_\theta\left[\left(\frac{p(x;\theta')}{p(x;\theta)} - 1\right)^2\right]}.$$

(b) Hence conclude that

$$\operatorname{var}_\theta(\delta(X)) \;\geq\; a^*/n^2,$$

where a^* solves the equation $2/(na) - \frac{e^{na}}{e^{na}-1} = 0$.

(c) The information inequality under i.i.d. sampling predicts scaling of the form $\operatorname{var}_\theta(\delta(X)) = \mathcal{O}(1/n)$. Explain why the result of (b) differs from this scaling.

(d) Prove that the estimator $\delta_a(X) = \min_i X_i - \frac{1}{n}$ is unbiased, and has variance $1/n^2$.

Problem 3.5
Let $X_1, \ldots, X_n$ be i.i.d samples from the $\text{Poi}(\lambda)$ distribution truncated on the left at 0 (i.e., a Poisson variate $Y \sim \text{Poi}(\lambda)$ conditioned on $Y \geq 1$). Show that the information inequality for any unbiased estimator of λ is

$$\frac{\lambda\,(1 - \exp^{-\lambda})^2}{n\;(1 - \exp(-\lambda) - \lambda\exp(-\lambda))}.$$