UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Problem Set 7
Fall 2014

Issued: Saturday, October 25 **Due:** Thursday, October 30 (beginning of class)

Problem 7.1
Method of moments estimation. Let $X_1, X_2, \ldots$ be i.i.d. observations from some family of distributions indexed by $\theta \in \Omega \subset \mathbb{R}$. Let $\bar{X}_n$ denote the average of the first n observations, and let $\mu(\theta) = E_\theta X_i$ and $\sigma^2(\theta) = \text{Var}_\theta(X_i)$. Assume that μ is strictly monotonic and continuously differentiable. The method of moments estimator $\hat{\theta}_n$ solves $\mu(\theta) = \bar{X}_n$. If $\mu'(\theta) \neq 0$, find the limiting distribution for $\sqrt{n}(\hat{\theta}_n - \theta)$.

Problem 7.2
Let $X_1, X_2, \ldots$ be i.i.d. from a uniform distribution on $(0, 1)$, and let $T_n \in [0, 1]$ be the unique solution of the equation

$$\sum_{i=1}^{n} t^{X_i} = \sum_{i=1}^{n} X_i^2.$$

(a) Show that $T_n \xrightarrow{p} c$ as $n \to \infty$, identifying the constant c.

(b) Find the limiting distribution for $\sqrt{n}(T_n - c)$ as $n \to \infty$.

Problem 7.3
Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. samples from a normal location model $N(\theta, 1)$, and that we are interested in estimating the quantity $1/\theta$. In order to do so, we use the estimator $\delta(X) = 1/\bar{X}_n$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean.

(a) Show that δ is asymptotically normal—viz.: $\sqrt{n}\left(1/\bar{X}_n - 1/\theta\right) \xrightarrow{d} N(0, 1/\theta^4)$.

(b) Show that the expectation $\mathbb{E}[1/\bar{X}_n]$ fails to exist for all n. Why does this not contradict the result of part (a)?

Problem 7.4
A variance stabilizing approach. Let $X_1, X_2, \ldots$ be i.i.d. from a Poisson distribution with mean θ, and let $\hat{\theta}_n = \bar{X}_n$ be the MLE of θ.

(a) Find a function $g : [0, \infty) \to \mathbb{R}$ such that

$$Z_n = \sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} N(0, 1).$$

(b) Find a $1 - \alpha$ asymptotic confidence interval for θ based on the approximate pivot Z_n.

Problem 7.5

Let $X_1, X_2, \ldots$ be i.i.d. from $N(\mu, \sigma^2)$. Suppose we know that σ is a known function of μ: $\sigma = g(\mu)$. Let $\hat{\mu}_n$ denote the MLE for μ under this assumption, based on $X_1, X_2, \ldots, X_n$.

(a) Give a $1 - \alpha$ asymptotic confidence interval for μ centered at $\hat{\mu}_n$. Hint: If $Z \sim N(0, 1)$, then $\text{Var}(Z^2) = 2$ and $\text{Cov}(Z, Z^2) = 0$.

(b) Compare the width of the asymptotic confidence interval in part (a) with the width of the t-confidence interval that would be appropriate if μ and σ were not functionally related. Specificially, show that the ratio of the two widths converges in probability as $n \to \infty$, identifying the limiting value. (The limit should be a function of μ.)