Stat210A: Theoretical Statistics **Lecture Date: November 4, 2014**

More on Hypothesis Testing, Local Power, and Unbiased Test

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Example 1. Suppose X follows a Poisson distribution, $p_\theta(x) = \theta e^{-\theta x}$. We want to test $H_0 : \theta = 1$ versus $H_1 : \theta = \theta_1 > 1$. Then we perform a *likelihood ration test*,

$$\phi(x) = \begin{cases} 1 & p_1(x)/p_0(x) = \theta_1 e^{-\theta_1 x}/e^{-x} > k \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, $\phi = 1$ when $x < \frac{\log(\theta_1/k)}{\theta_1 - 1} = k'$. k' can be set such that level

$$\alpha = P_0(X < k') = \int_0^{k'} e^{-x} dx = 1 - e^{-k'},$$

so that $k' = -\log(1-\alpha)$. Therefore,

$$\phi(x) = \begin{cases} 1 & x < -\log(1-\alpha) \\ 0 & \text{otherwise} \end{cases}$$

1 Uniformly Most Powerful (UMP) Tests

In short, a test ϕ^* is UMP if $\mathbb{E}_\theta \phi^* \geq \mathbb{E}_\theta \phi, \forall \theta \in \Omega$ for ϕ^* and ϕ having level α.

Definition 2. A family $p_\theta(x), \theta \in \Omega \subseteq \mathbb{R}$ has monotone likelihood ratios if there exists a statistic T such that whenever $\theta_1 < \theta_2$, the ratio $p_{\theta_2}(x)/p_{\theta_1}(x)$ is monotone (non-decreasing) in T.

Example 3. Exponential family random variable X has $p_\theta(x) = h(x)\exp(\eta(\theta)T(x) - B(\theta))$ with $\eta(\cdot)$ strictly increasing. Then for $\theta_2 > \theta_1$,

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \exp\{(\eta(\theta_2) - \eta(\theta_1))T(x) + B(\theta_1) - B(\theta_2)\}$$

is non-decreasing in $T(x)$ and therefore has monotone likelihood ratios.

Theorem 4. *(12.9 in Keener) Suppose that p_θ has monotone likelihood ratios, then*

1 *The test ϕ^* given by*

$$\phi^*(x) = \begin{cases} 1 & T(x) > c \\ \gamma & T(x) = c \\ 0 & T(x) < c \end{cases}$$

is UMP for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

2 *The power function $\beta(\theta) = \mathbb{E}_\theta \phi^*$ is non-decreasing and strictly increasing for $\beta(\theta) \in (0, 1)$.*

3 *If $\theta_1 \leq \theta_0$, then ϕ^* minimize $\mathbb{E}_{\theta_1} \phi$ among all tests such that $\mathbb{E}_{\theta_0} \phi = \mathbb{E}_{\theta_0} \phi^* = \alpha$.*

Example 5. X_i's are iid with $U(0,\theta)$. $M(x) = \min\{x_1,\ldots,x_n\}$ and $T(x) = \max\{x_1,\ldots,x_n\}$.

$$p_\theta(x) = \begin{cases} 1/\theta^n & M(x) > 0, T(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

If $\theta_2 > \theta_1$, then

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \begin{cases} \theta_1^n/\theta_2^n & T(x) < \theta_1 \\ 0 & \text{otherwise} \end{cases}$$

. Thus, it has monotone likelihood ratios. $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$. Follow by the Theorem,

$$\phi^*(x) = \begin{cases} 1 & T(x) \ge c \\ 0 & T(x) < c \end{cases}$$

is UMP. The level is $P_1(T \ge c) = 1 - c^n$ is set to α. Thus, $c = (1-\alpha)^{1/n}$. The power of the test

$$\beta_\phi(\theta) = P_\theta(T \ge c) = \begin{cases} 0 & \theta < c \\ 1 - (\frac{c}{\theta})^n = 1 - \frac{1-\alpha}{\theta^n} & \text{otherwise} \end{cases}$$

2 Local Power

Local Power is a popular concept. We first assume enough regularity such that

$$\beta'(\theta) = \int \phi(x)\frac{\partial p_\theta(x)}{\partial\theta}\mu(dx).$$

A test ϕ^* is locally most powerful (LMP) for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ if it maximizes $\beta'_\phi(\theta_0)$ among all tests with level α. The major characterization for LMP test is that the test should maximize $\int \phi(x)\frac{\partial \log p_\theta(x)}{\partial\theta}p_\theta(x)\mu(dx)$ subject to $\int \phi p_\theta d\mu = \alpha$. Therefore, the test is in the form

$$\phi^*(x) = \begin{cases} 1 & \frac{\partial \log p_\theta(x)}{\partial\theta} > k \\ 0 & \frac{\partial \log p_\theta(x)}{\partial\theta} < k \end{cases}$$

Example 6. X_i's iid has density $f_\theta(x)$, then $\log p_\theta(x) = \sum_i \log f_\theta(x_i)$. From CLT,

$$\frac{1}{\sqrt{n}}\frac{\partial \log p_\theta(x)}{\partial\theta} \Rightarrow N(0, I(\theta)).$$

Level of ϕ^* is approximately

$$p_{\theta_0}\left(\frac{1}{\sqrt{n}}\frac{\partial \log p_\theta(x)}{\partial\theta} < \frac{k}{\sqrt{n}}\right) \approx 1 - \Phi\left(\frac{k}{\sqrt{nI(\theta_0)}}\right),$$

which is α if $k = z_\alpha\sqrt{nI(\theta_0)}$.

3 Unbiased Tests

Definition 7. A test is *unbiased* if the test has power $\beta_\phi(\theta) \le \alpha, \forall\theta \in \Omega_0$ and $\beta_\phi(\theta) \ge \alpha, \forall\theta \in \Omega_1$.

Theorem 8. *(12.26 in Keener) For $\alpha \in (0,1)$, let $\theta_0 \in \Omega^0$, X is exponential family random variable with η differentiable and $0 < \eta'(\theta_0) < \infty$. Then there exist a two-sided, level α test ϕ^* with $\beta'_{\phi^*}(\theta_0) = 0$ and any such test is UMP for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$ among unbiased tests at level α.*