Problem Set 1: Complex Analysis-I

Kreyszig Section No.s	Topics
13.1	**Complex Numbers and Their Geometric Representation**
13.2	**Polar Form of Complex Numbers, Powers and Roots**
13.5	**Exponential Function**
13.6	**Trigonometric and Hyperbolic Functions, Eulers Formula**
13.7	**Logarithms. General Powers. Principal Values**

1 Complex Numbers and Their Geometric Representation

Complex Numbers

Plotting complex numbers on the complex plane

1. Let $z_1 = -2 + 5i, z_2 = 3 - i$. find solutions of the following questions and write the answer in the form of $x + iy$:

   (a) $z_1 z_2, (\bar{z_1 z_2})$;

   (b) $Re(z_1^2), Re(z_1)^2$;

   (c) $(z_1 + z_2)(z_1 - z_2), z_1^2 - z_2^2$;

   (d) $\bar{z}_1/\bar{z}_2, (\bar{z_1/z_2})$;

2. Let $z = x + iy$, find solutions of the following questions in terms of x and y:

   (a) $Im(1/z), Im(1/z^2)$;

   (b) $Re[(1 + i)^{16} z^2]$;

   (c) $Im(1/\bar{z}^2)$;

3. Plot the solutions in Question 1 and 2 in the complex plane. Calculate the value or formula in Q1 and Q2 by Matlab (hint: try function "real", "imag") or Wolfram Alpha (hint: just enter the complex number; use "re(z)" and "im(z)" to get the real and imaginary parts)

2 Polar Form of Complex Numbers. Powers and Roots

Example: Complex roots for a quadratic

Euler's Formula and Euler's Identity

1. Represent in polar form and graph in the complex plane.
(Hints: Try to use *abs()* and *angle()* in Matlab; *abs()* and *arg()* in Wolfram Alpha)

   (a) (*) $1+i$;
   (b) (*) $2i$;
   (c) (*) -5;
   (d) (**) $\frac{\sqrt{2}+i/3}{-\sqrt{8}-2i/3}$;

2. Determine the principal value of the argument and graph it.

   (a) (*) $-1+i$;
   (b) (*) $-\pi-\pi i$;
   (c) (**) $(1+i)^{20}$;

3. Find and graph all roots in the complex plane.

   (a) (*) $\sqrt[3]{1+i}$;
   (b) (*) $\sqrt[3]{216}$;
   (c) (*) $\sqrt[4]{i}$;
   (d) (*) $\sqrt[5]{-1}$;

4. (*) Verify **triangle inequality** for $z_1 = 3+i$ and $z_2 = -2+4i$.

5. (**) **Parallelogram equality**. Prove and explain the name.

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right) \tag{1}$$

3 Exponential Function

Exponential form to find complex roots

1. **Function Values.** Find e^z in the form $u+iv$ and $|e^z|$ if z equals

   (a) (*) $2\pi i\,(1+i)$;
   (b) (*) $2+3\pi i$;
   (c) (*) $\sqrt{2}+\frac{1}{2}\pi i$;

2. **Polar Form.** Write in exponential form $z = re^{i\theta}$

   (a) (*) $4+3i$;
   (b) (*) -6;
   (c) (**) $\sqrt{i}$;

3. **Real and Imaginary Parts.** Find Re and Im of

   (a) (*) $e^{-\pi z}$;
   (b) (*) $\exp\left(z^2\right)$;

(c) (**) $e^{1/z}$;

4. **Equations.** Find all solutions and graph some of them in the complex plane.

   (a) (*) $e^z = 1$;
   (b) (*) $e^z = 4 + 3i$;
   (c) (**) $e^z = 0$;
   (d) (*) $e^z = -2$;

4 Trigonometric and Hyperbolic Functions. Euleros Formula

Hyperbolic Trig Function Inspiration

Trigonometric Identities

1. Find the values of following functions, in the form $u + iv$,

   (a) (*) $\sin 2\pi i$;
   (b) (*) $\cos(-2 - i)$;
   (c) (*) $\cos\left[\frac{1}{2}\pi(1 + i)\right]$.

2. Show that,

   (a) (*) $\cos(-z) = \cos z$;
   (b) (*) $\sin(-z) = -\sin z$;
   (c) (*) $\operatorname{Re} \sin z$ is harmonic;
   (d) (*) $\operatorname{Im} \cos z$ is harmonic;
   (e) (*) $\sin z_1 \cos z_2 = \frac{1}{2}\left[\sin(z_1 + z_2) + \sin(z_1 - z_2)\right]$;
   (f) (**) $\sin nx = 2\cos x \sin(n-1)x - \sin(n-2)x$;
   (g) (**) $\tan 2z = \frac{2\tan z}{1 - \tan^2 z}$.

3. Find the solutions,

   (a) (*) $\sin(-z) = 100$;
   (b) (**) $\cos(2z) = -33$.

4. Find the values of following functions, in the form $u + iv$,

   (a) (*) $\sinh(3 + \pi i)$;
   (b) (*) $\cosh(1 - \pi i)$;
   (c) (*) $\tanh(2 + 4\pi i)$.

5. Show that,

   (a) (*) $\cosh z = \cosh x \cos y + i \sinh x \sin y$;
   (b) (*) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$;

(c) (**) $|\sinh y| \leq |\cos z| \leq \cosh y$;
(d) (**) $|\sinh y| \leq |\sin z| \leq \cosh y$;
(e) (**) the complex cosine and sine are not bounded in the whole complex plane.

6. Find the solutions,
   (a) (**) $\sinh z = 0$;
   (b) (**) $\sinh z = i$.

5 Logarithms. General Powers. Principal Values

Powers of complex numbers

i as the Principal Root of -1 (a little technical)

1. (*) What is the imaginary part of i^i?
2. Find the principal values,
   (a) (*) $\mathrm{Ln}\,(4 + 4i)$;
   (b) (*) $\mathrm{Ln}\,(ei)$;
   (c) (*) $\mathrm{Ln}\,(0.6 + 0.8i)$.
3. Find all values,
   (a) (*) $\ln\left(e^i\right)$;
   (b) (*) $\ln\left(i^2\right)$;
   (c) (*) $\ln\,(2i)$.
4. Compare the values of the following pairs,
   (a) (*) $\ln z^2$, $2\ln z$;
   (b) (*) $\ln\sqrt{z}$, $\frac{1}{2}\ln z$.
5. Find the principal value,
   (a) (*) $(1 + i)^{1-i}$;
   (b) (*) $(1 - i)^{1+i}$.
   (c) (*) $(i)^{i/2}$;
   (d) (*) $(3 + 4i)^{\frac{1}{3}}$.
6. (***) Let z be defined as $e^{i2\pi/n}$. Prove that $1 + z + z^2 + ... + z^{n-1}$ equals 0.