Solution 1: Complex Numbers and Functions

1 Complex Numbers and Their Geometric Representation

1. (a) $-1 + 17i$, $-1 - 17i$
   (b) -117, 4;
   (c) $-29 - 14i$, $-29 - 14i$;
   (d) $-11/10 - 13/10i$, $-1/10 + 13/10i$.
2. (a) $-y/(x^2 + y^2)$,
   (b) $2^{15}(x^2 - y^2 - 2xy)$
   (c) $2xy/((x^2 - y^2)^2 + 4x^2y^2)$.
3. (omited)

2 Polar Form of Complex Numbers. Powers and Roots

1. Represent in polar form and graph in the complex plane.
   (a) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
   (b) $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
   (c) $5\left(\cos\pi + i\sin\pi\right)$
   (d) $\frac{1}{2}\left(\cos\pi + i\sin\pi\right)$
2. Determine the principal value of the argument and graph it.
   (a) $3\pi/4$;
   (b) $-3\pi/4$;
   (c) π;
3. Find and graph all roots in the complex plane.
   (a) $\sqrt[6]{2}\left(\cos\frac{k\pi}{12} + i\sin\frac{k\pi}{12}\right)$, $k = 1, 9, 17$;
   (b) 6, $-3 \pm 3\sqrt{3}i$;
   (c) $\cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{k\pi}{2}\right)$, $k = 0, 1, 2, 3$;
   (d) $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$, -1;

3 Exponential Function

1. **Function Values.** Find e^z in the form $u + iv$ and $|e^z|$ if z equals
   (a) $e^{-2\pi}$, $e^{-2\pi}$;
   (b) $-e^2$, e^2;
   (c) $e^{\sqrt{2}i}$, $e^{\sqrt{2}}$;

2. **Polar Form.** Write in exponential form $z = re^{i\theta}$
   (a) $5e^{i\arctan(3/4)}$;
   (b) $6e^{i\pi}$;
   (c) $e^{i\pi/4}$;

3. **Real and Imaginary Parts.** Find Re and Im of
   (a) $e^{-\pi x}\cos(\pi y), -e^{-\pi x}\sin(\pi y)$;
   (b) $\exp(x^2 - y^2)\cos(2xy)$, $\exp(x^2 - y^2)\sin(2xy)$;
   (c) $e^{\frac{x}{x^2+y^2}}\cos\left(\frac{y}{x^2+y^2}\right)$, $-e^{\frac{x}{x^2+y^2}}\sin\left(\frac{y}{x^2+y^2}\right)$;

4. **Equations.** Find all solutions and graph some of them in the complex plane.
   (a) $z = 2n\pi i$ where n is an arbitrary integer;
   (b) $z = \ln 5 + i\arctan\frac{3}{4} + 2n\pi i$ where n is an arbitrary integer;
   (c) no solution;
   (d) $z = \ln 2 + \pi i + 2n\pi i$ where n is an arbitrary integer;

4 Trigonometric and Hyperbolic Functions. Euler's Formula

1. (a) $\sinh(2\pi)\, i$;
   (b) $\cosh 1 \cos 2 - i \sinh 1 \sin 2$;
   (c) $-i\sinh\left(\frac{\pi}{2}\right)$.

2. (a) omitted;
   (b) omitted;
   (c) Firstly, we can get $\text{Re}\sin z = \sin x\cosh y$;
   Then we have $\frac{\partial^2 \text{Re}\sin z}{\partial x^2} = -\sin x\cosh y$, $\frac{\partial^2 \text{Re}\sin z}{\partial y^2} = \sin x\cosh y$.
   Finally, $\frac{\partial^2 \text{Re}\sin z}{\partial x^2} + \frac{\partial^2 \text{Re}\sin z}{\partial y^2} = 0$, which means $\text{Re}\sin z$ is harmonic;
   (d) omitted;
   (e) omitted;
   (f) Firstly, we write down $\sin nx = \sin((n-1)x + x) = \sin(n-1)x\cos x + \sin x\cos(n-1)x$;
   Then we write down $\sin(n-2)x = \sin((n-1)x - x) = \sin(n-1)x\cos x - \sin x\cos(n-1)x$;
   So we have $\sin nx + \sin(n-2)x = 2\sin(n-1)x\cos x$.
   (g) $\frac{2\tan z}{1-\tan^2 z} = \frac{2\frac{\sin z}{\cos z}}{1-\left(\frac{\sin z}{\cos z}\right)^2} = \frac{2\sin z\cos z}{\cos^2 z - \sin^2 z} = \frac{2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iz}+e^{-iz}}{2}\right)}{\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2} = \frac{\sin 2z}{\cos 2z} = \tan 2z$

3. (a) $z = 1.57 - 5.30i \pm 2k\pi$ or $-1.57 + 5.30i \pm 2k\pi$ with $k \in \{0, 1, 2, \ldots\}$;
   (b) $z = 1.57 + 2.09i \pm 2k\pi$ or $1.57 - 2.09i \pm 2k\pi$ or $-1.57 + 2.09i \pm 2k\pi$ or $-1.57 - 2.09i \pm 2k\pi$ with $k \in \{0, 1, 2, \ldots\}$.
4. (a) $-\sinh 3$;
   (b) $\cosh 1$;
   (c) $\frac{\sinh 4}{\cosh 4 + 1}$.
5. (a) omitted;
   (b) omitted;
   (c) The proof of right inequality: $|\cos z| = \left|\frac{e^{zi}+e^{-zi}}{2}\right| = \left|\frac{e^{xi}e^{-y}+e^{-xi}e^{y}}{2}\right| \leq \frac{|e^{xi}||e^{-y}|}{2} + \frac{|e^{-xi}||e^{y}|}{2} = \frac{|e^{y}|+|e^{-y}|}{2} = \cosh y$;
   The proof of left inequality: $|\cos(x+iy)| = |\cos x \cosh y - i \sin x \sinh y| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} = \sqrt{(1-\sin^2 x)(1+\sinh^2 y) + \sin^2 x \sinh^2 y} = \sqrt{\cos^2 x + \sinh^2 y} \geq |\sinh y|$.
   (d) omitted;
   (e) We first prove $|\sinh y| > |y|$:
   Construct $f(y) = \sinh y - y$, we have $f'(y) = \cosh y - 1 > 0$ for $y > 0$. Therefore, $f(y)$ is an increasing function of y for $y > 0$. Also we have $f(0) = 0$. We conclude that $f(y) > 0$, ie. $\sinh y - y > 0$ for all $y > 0$. Since $\sinh y$ is odd. We get $|\sinh y| \geq |y|$ for any $y \in \mathbb{R}$.
   Together with (c)'s result, we have $|\cos z| \geq |y|$. Since $|y|$ approximates infinity as $y \to \infty$, $|\cos z|$ is unbounded in the complex plane.
6. (a) $z = \pm k\pi i$ with $k \in \{0, 1, 2, \ldots\}$;
   (b) omitted.

5 Logarithms. General Powers. Principal Values

1. 0;
2. (a) $1.73 + 0.79i$;
   (b) $1 + 1.57i$;
   (c) 0.93.
3. (a) $i + 2k\pi i$ with $k \in \{0, 1, 2, \ldots\}$;
   (b) $\pi i + 2k\pi i$;
   (c) $\ln 2 + \frac{\pi}{2}i + 2k\pi i$ with $k \in \{0, 1, 2, \ldots\}$.
4. (a) omitted;
   (b) omitted.
5. (a) $e^{\ln\sqrt{2}+\frac{\pi}{4}}\left(\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i \sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right)$;
   (b) omitted;

(c) omitted;

(d) $e^{\frac{1}{3}\ln 5}\left(\cos\left(\frac{1}{3}\arctan\frac{4}{3}\right)+i\sin\left(\frac{1}{3}\arctan\frac{4}{3}\right)\right)$.

6. (Here the question needs to claim that n is an integer which is larger than 1.)
Denote $1+z+z^2+\ldots+z^{n-1}$ by $S(z)$. We have $zS(z)-S(z)=z^n-1$, so $S(z)=\frac{z^n-1}{z-1}$. Replace z by $e^{\frac{2\pi}{n}i}$, we have $S(z)=\frac{e^{2\pi i}-1}{e^{\frac{2\pi}{n}i}-1}=0$.