Problem Bank 2: Complex Differentiation and Integration

Kreyszig Section No.s	Topics	Slide No.s
13.3	**Derivatives. Analytic Functions**	**13-24**
13.4	**CauchyRiemann Equations. Laplaces Equation**	
14.1	**Line Integral in the Complex Plane**	**34-47**
14.2	**Cauchys Integral Theorem**	
14.3	**Cauchys Integral Formula**	
14.4	**Derivatives of Analytic Functions**	

1 Derivative. Analytic Function

derivatives example:

derivatives example 2

1. Determine and skectch or graph the sets in the complex plane given by

   (a) $|z + 1 - 2i| \leq \frac{1}{4}$;

   (b) $-\pi < \mathrm{Im}(z) < \pi$;

   (c) $\mathrm{Re}(z) \leq 1$;

   (d) $|z + i| \geq |z - i|$;

2. Find $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ and their values at given point z:

   (a) $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$;

   (b) $f(z) = (z - 1)(z + 1)$ at $2i$;

3. Find the value of the derivative of

   (a) $(z - i)/(z + i)$ at i;

   (b) $(z - 2i)^3$ at $5 + 2i$;

   (c) $i(1 - z)^n$ at 0;

   (d) $z^3/(z - i)^3$ at $-i$;

4. (**)

   (a) Prove the following two statements are equivalent

      i. $\lim_{z\to z_0} f(z) = l$;

      ii. $\lim_{z\to z_0} \mathrm{Re}(f(z)) = \mathrm{Re}(l)$ and $\lim_{z\to z_0} \mathrm{Im}(f(z)) = \mathrm{Im}(l)$;

   (b) If $\lim_{z\to z_0} f(x)$ exists, show that this limit is unique. (Hint: Proofs by Contradiction, if it is not unique, then...)
   (c) If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.
   (d) Show that $f(z) = |z|^2$ is nowhere analytic. (Hint: first show it is only differentiable at $z = 0$ hence it is nowhere analytic)

2 Cauchy-Riemann Equations. Laplace's Equation

caichy-riemann example
harmonic function example

1. (*)knowing $u_x = v_y$ and $u_y = -v_x$, prove $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$
2. (*)Are the following functions analytic? Use Cauchy-Riemann Equations.
   (a) $f(z) = iz\bar{z}$;
   (b) $f(z) = e^x(\cos y - i\sin y)$;
   (c) $f(z) = \text{Re}(z^2) - i\text{Im}(z^2)$;
   (d) $f(z) = \ln|z| + i\text{Arg}(z)$.
3. (**)Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$
   (a) $u = x^3 + y^3$;
   (b) $u = xy$.
   (c) $u = e^{-x}\sin 2y$.
4. (**)Determine a and b so that the given function is harmonic and find a harmonic conjugate.
   (a) $u = e^{-\pi x}\cos ay$;
   (b) $u = ax^3 + bxy$.
   (c) $u = \cosh ax \cos y$.

3 Line Integral in the Complex Plane

integration basics

1. Find the path and sketch it:
   (a) $z(t) = \left(1 + \frac{1}{2}i\right)t$ $(2 \le t \le 5)$;
   (b) $z(t) = t + 2it^2$ $(1 \le t \le 2)$;
   (c) $z(t) = 3 - i + \sqrt{10}e^{-it}$ $(0 \le t \le 2\pi)$;
   (d) $z(t) = 2 + 4e^{\frac{\pi}{2}it}$ $(0 \le t \le 2)$.
2. Find a parameter representation and sketch the path:

(a) segment from $(-1,1)$ to $(1,3)$;
(b) upper half of $|z-2+i|=2$ from $(4,-1)$ to $(0,-1)$;
(c) $x^2-4y^2=4$, the branch through $(2,0)$;
(d) $|z+a+bi|=r$, clockwise.

3. Integrate by the first method or state why it does not apply and use the second method:

   (a) $\int_C \mathrm{Re} z dz$, C is the shortest path from $1+i$ to $3+3i$;
   (b) $\int_C e^z dz$, C is the shortest path from πi to $2\pi i$;
   (c) $\int_C z\exp\left(z^2\right)dz$, C is from 1 along the axes to i;
   (d) $\int_C \sec^2 z dz$, any path from $\frac{\pi}{4}$ to $\frac{\pi i}{4}$;
   (e) $\int_C \mathrm{Im}\left(z^2\right)dz$, counterclockwise around the triangle with vertices $0,1,i$.

4. Verify $\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz$ for $f(z)=z^2$, where C is the segment from $-1-i$ to $1+i$.

4 Cauchy's Integral Theorem

1. Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies:

   (a) $f(z)=\exp\left(-z^2\right)$;
   (b) $f(z)=\frac{1}{2z-1}$;
   (c) $f(z)=\frac{1}{z^4-1.1}$;
   (d) $f(z)=\mathrm{Im} z$;
   (e) $f(z)=\frac{1}{|z|^2}$.

2. Evaluate the integral, does Cauchy's theorem apply?

   (a) $\oint_C \mathrm{Ln}(1-z)\,dz$, C is the boundary of the parallelogram with vertices $\pm i$, $\pm(1+i)$;
   (b) $\oint_C \frac{e^z}{z}dz$, C consists of $|z|=2$ counterclockwise and $|z|=1$ clockwise;
   (c) $\oint_C \frac{\cos z}{z}dz$, C consists of $|z|=1$ counterclockwise and $|z|=3$ clockwise;
   (d) $\oint_C \frac{\sin z}{z+2iz}dz$, C is $|z-4-2i|=5.5$ clockwise.

3. If the integral of a function over the unit circle equals 2 and over the circle of radius 3 equals 6, can the function be analytic everywhere in the annulus $1<|z|<3$?

5 Cauchy's Integral Formula

cauchy iintegral formula

1. Integrate $z^2/\left(z^2-1\right)$ by Cauchyos formula counterclockwise around the circle.
   (a) (*) $|z+1|=1$
   (b) (*) $|z+i|=1.4$
2. Integrate the given function around the unit circle.
   (a) (*) $(\cos 3z)\,/\,(6z)$
   (b) (*) $z^3/\,(2z-1)$
3. Integrate counterclockwise or as indicated. Show the details.
   (a) (*) $\oint_C \frac{dz}{z^2+4}, \quad C: 4x^2+(y-2)^2=4$
   (b) (*) $\oint_C \frac{z+2}{z-2}dz, \quad C: |z-1|=2$
   (c) (*) $\oint_C \frac{\mathrm{Ln}\,(z+1)}{z^2+1}dz, \quad C: |z-i|=1.4$
   (d) (*) $\oint_C \frac{\exp z^2)}{z^2\,(z-1-i)}dz, \quad C$ consists of $|z|=2$ counter-clockwise and $|z|=1$ clockwise.

6 Derivatives of Analytic Functions

1. Integrate counterclockwise around the unit circle.
   (a) (*) $\oint_C \frac{\sin z}{z^4}dz$
   (b) (*) $\oint_C \frac{e^z}{z^n}dz, \quad n=1,2,\ldots$
   (c) (*) $\oint_C \frac{\cosh 2z}{\left(z-\frac{1}{2}\right)^4}dz$
   (d) (*) $\oint_C \frac{\cos 2z}{z^{2n+1}}dz, \quad n=1,2,\ldots$
2. Integrate. Show the details. Hint. Begin by sketching the contour. Why?
   (a) (*) $\oint_C \frac{\tan \pi z}{z^2}dz, \quad C$ the ellipse $16x^2+y^2=1$ clockwise.
   (b) (*) $\oint_C \frac{(1+z)\sin z}{(2z-1)^2}dz, \quad C: |z-i|=2$ counterclockwise.
   (c) (*) $\oint_C \frac{\mathrm{Ln}\,z}{(z-2)^2}dz, \quad C: |z-3|=2$ counterclockwise.

(d) (*) $\oint_C \frac{\cosh 4z}{(z-4)^3} dz$, C consists of $|z| = 6$ counterclockwise and $|z-3| = 2$ clockwise.

(e) (*) $\oint_C \frac{e^{-z} \sin z}{(z-4)^3} dz$, C consists of $|z| = 5$ counterclockwise and $|z-3| = \frac{3}{2}$ clockwise.

(f) (*) $\oint_C \frac{e^{3z}}{(4z - \pi i)^3} dz$, $C : |z| = 1$ counterclockwise.