Problem Bank 4: First-Order ODEs

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1 Basic Concepts. Modeling

1. Solve the ODE by integration or by remembering a differentiation formula.

   (a) $y' + 2\sin 2\pi x = 0$;

   (b) $y' + xe^{-x^2/2} = 0$;

   (c) $y' = -1.5y$;

   (d) $y'' = -y$;

   (e) $y''' = e^{-0.2x}$

2. (1) Verify that y is a solution of the ODE. (2) Determine from y the particular solution of the IVP. (3) Graph the solution of the initial value problem (IVP).

   (a) $y' + 5xy = 0$, $y = ce^{-2.5x^2}$, $y(0) = \pi$;

   (b) $yy' = 4x$, $y^2 - 4x^2 = c(y > 0)$, $y(1) = 4$;

   (c) $y' \tan x = 2y - 8$, $y = c\sin^2 x + 4$, $y(\frac{1}{2}\pi) = 0$;

3. **Half-life**. Radium ${}^{224}_{88}R_a$ has a half-life of about 3.6 days.

   (a) Given 1 gram, how much will still be present after 1 day?

   (b) After 1 year?

4. **Exponential decay. Subsonic flight.** The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near $35,000$ ft. Find the air pressure $y(x)$ at this height. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure. At $18,000$ ft it is half its value $y_0 = y(0)$ at sea level. *Hint.* Remember from calculus that if $y = e^{kx}$, then $y' = ke^{kx} = ky$ Can you see without calculation that the answer should be close to $y_0/4$?

2 Geometric Meaning of $y' = f(x,y)$. Direction Fields, Euler's Method

1. Graph a direction field (by Matlab or by hand). In the field graph several solution curves by hand, particularly those passing through the given points (x,y).

   (a) $yy' + 4x = 0, \quad (1,1), (0,2);$

   (b) $y' = x - 1/y, \quad (1, \frac{1}{2});$

   (c) $y' = -2xy, \quad (0, \frac{1}{2}),\ (0,1),\ (0,2);$

3 Separable ODEs. Modeling

1. Find a general solution. Show the steps of derivation. Check your answer by substitution.

   (a) $y^3y' + x^3 = 0;$

   (b) $y' \sin 2\pi x = \pi y \cos 2\pi x;$

   (c) $yy' + 36x = 0;$

   (d) $y' = e^{2x-1}y^2;$

   (e) $y' = (y+4x)^2 \quad$ (Set $y + 4x = v$);

   (f) $xy' = x + y \quad$ (Set $y/x = u$)

2. Solve the IVP. Show the steps of derivation, beginning with the general solution.

   (a) $xy' + y = 0, \quad y(4) = 6;$

   (b) $y' = 1 + 4y^2, \quad y(1) = 0;$

   (c) $dr/dt = -2tr, \quad r(0) = r_0;$

   (d) $y' = -4x/y, \quad y(2) = 3;$

   (e) $y' = (x+y-2)^2, \quad y(0) = 2 \quad$ (Set $v = x + y - 2$);

   (f) $xy' = y + 3x^4\cos^2(y/x), \quad y(1) = 0 \quad$ (Set $y/x = u$);

3. **An population model.**(a) If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time. (b) What is the limiting situation for increasing time? Interpret it.

4. **Gompertz growth in tumors**. The Gompertz model is $y' = -Ay\ln y$ $(A > 0)$, where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.

5. **Family of Curves**. A family of curves can often be characterized as the general solution of $y' = f(x,y)$.

   (a) Show that for the circles with center at the origin, we get $y' = -x/y$.

   (b) Graph some of the hyperbolas $xy = c$. Find an ODE for them.

   (c) Find an ODE for the straight lines through the origin.

4 Exact ODEs. Integrating Factors

1. Test for exactness. If exact, solve. If not, use an integrating factor as given or obtained by inspection or by the theorems in the text. Also, if an initial condition is given, find the corresponding particular solution.

   (a) $x^3 dx + y^3 dy = 0;$

   (b) $e^{3\theta}(dr + 3r d\theta) = 0;$

   (c) $3(y+1)\,dx = 2x dy, \quad (y+1)\,x^{-4};$

   (d) $e^x(\cos y dx - \sin y dy) = 0;$

   (e) $y dx + [y + \tan(x+y)]\,dy = 0, \quad \cos(x+y);$

   (f) $(2xy dx + dy)e^{x^2} = 0.\ \ y(0) = 2$

   (g) $(a+1)\,y dx + (b+1)\,x dy = 0, \quad y(1) = 1, \quad F = x^a y^b;$

2. Under what conditions for the constants $a,\ b,\ k,\ l$ is $(ax+by)\,dx + (kx+ly)\,dy = 0$ exact? Solve the exact ODE.

3. **Solution by Several Methods**. Show this as indicated. Compare the amount of work.

   (a) $e^y(\sinh x dx + \cosh x dy) = 0$ as an exact ODE and by separation.

   (b) $(1+2x)\cos y dx + dy/\cos y = 0$ by Theorem 2 and by separation.

   (c) $(x^2+y^2)dx - 2xy dy = 0$ by Theorem 1 and 2 and by separation.

   (d) $3x^2 y dx + 4x^3 dy = 0$ by Theorem 1 and 2 and by separation.

5 Linear ODEs. Bernoulli Equation. Population Dynamics

1. Find the general solution. If an initial condition is given, find also the corresponding particular solution and graph or sketch it. (Show the details of your work.)

   (a) $y' = 2y - 4x;$

   (b) $y' + 2y = 4\cos 2x, \quad y\left(\frac{1}{4}\pi\right) = 3;$

   (c) $y' + y\tan x = e^{-0.01x}\cos x, \quad y(0) = 0;$

   (d) $y'\cos x + (3y-1)\sec x = 0, \quad y(\frac{1}{4}\pi) = 4/3;$

   (e) $xy' + 4y = 8x^4, \quad y(1) = 2;$

2. Show that non-homogeneous linear ODEs (1) and homogeneous linear ODEs (2) have the following property:

   (a) $y = 0$ (that is, for all x, also written $y \equiv 0$) is a solution of (2) [not of (1) if $r(x) \neq 0$!], called the **trivial solution**.

   (b) The difference of two solutions of (1) is a solution of (2).

   (c) If y_1 and y_2 are solutions of $y_1' + py_1 = r_1$ and $y_2' + py_2 = r_2$, respectively (with the same p!), what can you say about the sum $y_1 + y_2$?

3. Using a method of this section or separating variables, find the general solution of nonlinear ODEs. If an initial condition is given, find also the particular solution and sketch or graph it.

(a) $y' + y = y^2$, $y(0) = -\frac{1}{3}$;

(b) $y' + y = -x/y$;

(c) $y' = (\tan y)/(x-1)$, $y(0) = \frac{1}{2}\pi$;

(d) $2xyy' + (x-1)y^2 = x^2e^x$ (Set $y^2 = z$);

4. **Heating and cooling of a building**. Heating and cooling of a building can be modeled by the ODE
$$T' = k_1(T - T_a) + k_2(T - T_\omega) + P$$
where $T = T(t)$ is the temperature in the building at time t, T_a the outside temperature, T_ω the temperature wanted in the building, and P the rate of increase of T due to machines and people in the building, and k_1 and k_2 are (negative) constants. Solve this ODE, assuming $P = const$, $T_\omega = const$, and T_a varying sinusoidally over 24 hours, say, $T_a = A - C\cos(2\pi/24)t$. Discuss the effect of each term of the equation on the solution.

5. **Epidemics**. A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and non-infected persons, who are assumed to move freely among each other. Set up the model. Find the equilibrium solutions and indicate their stability or instability. Solve the ODE. Find the limit of the proportion of infected persons as and explain what it means.

6 Existence and Uniqueness of Solutions for Initial Value Problems

1. **Existence**? Does the initial value problem $(x-2)y' = y$, $y(2) = 1$ have a solution? Does your result contradict our present theorems?

2. **Change of initial condition**. What happens in Prob.1 if you replace $y(2) = 1$ with $y(2) = k$?

3. **Lipschitz condition**. Show that for a linear ODE $y' + p(x)y = r(x)$ with continuous p and r in $|x - x_0| \leqq a$ a Lipschitz condition holds. This is remarkable because it means that for a *linear* ODE the continuity of $f(x, y)$ guarantees not only the existence but also the uniqueness of the solution of an initial value problem.

4. **Three possible cases**. Find all initial conditions such that $(x^2 - x)y' = (2x-1)y$ has no solution, precisely one solution, and more than one solution.