Solution 5: Second-Order and Higher Order Linear ODEs

1 Constant coefficient ODEs

1. Find general solutions of the following equations. Show the details of your calculation:

   (a) $y'' - 2y' - 3y = 0$
   $\lambda^2 - 2\lambda - 3 = 0$. $\lambda_1 = -1, \lambda_2 = 3$
   Complete solution: $y = c_1 e^{-x} + c_2 e^{3x}$

   (b) $y'' - 2y' + 5y = 0$
   $\lambda^2 - 2\lambda + 5 = 0$, $\lambda_{1,2} = 1 \pm 2i$
   Complete solution: $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$

   (c) $y'' + 6y' + 13y = 0$
   $\lambda^2 + 6\lambda + 13 = 0$, $\lambda_{1,2} = -3 \pm 2i$
   Complete solution: $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$

   (d) $4\frac{d^2x}{dt^2} - 20\frac{dx}{dt} + 25x = 0$
   $4\lambda^2 - 20\lambda + 25 = 0$, $\lambda_1 = \lambda_2 = \frac{5}{2}$
   Complete solution: $x = (c_1 + c_2 t)e^{\frac{5}{2}t}$

2. Solve the ODE with the given initial values:

   (a) $y'' - 4y' + 3y = 0$, $y(0) = 6, y'(0) = 10$
   $\lambda^2 - 4\lambda + 3 = 0$, $\lambda_1 = 1, \lambda_2 = 3$
   Complete solution: $y = c_1 e^{x} + c_2 e^{3x}$
   and we get: $y' = c_1 e^{x} + 3c_2 e^{3x}$
   Thus $\begin{cases} c_1 + c_2 = 6 \\ c_1 + 3c_2 = 10 \end{cases} \rightsquigarrow \begin{matrix} c_1 = 4 \\ c_2 = 2 \end{matrix}$
   Final result: $y = 4e^{x} + 2e^{3x}$

   (b) $4y'' + 4y' + y = 0$, $y(0) = 2, y'(0) = 0$
   $4\lambda^2 + 4\lambda + 1 = 0$, $\lambda_{1,2} = -\frac{1}{2}$
   Complete solution: $y = (c_1 + c_2 x)e^{-\frac{x}{2}}$
   and we get: $y' = (-\frac{c_1}{2} + c_2 - \frac{c_2}{2}x)e^{-\frac{x}{2}}$
   Thus $\begin{cases} c_1 = 2 \\ -\dfrac{c_1}{2} + c_2 = 0 \end{cases} \rightsquigarrow \begin{matrix} c_1 = 2 \\ c_2 = 1 \end{matrix}$
   Final result: $y = (2 + x)e^{-\frac{x}{2}}$

   (c) $y'' - 3y' - 4y = 0$, $y(0) = 0, y'(0) = -5$
   $\lambda^2 - 3\lambda - 4 = 0$, $\lambda_1 = -1, \lambda_2 = 4$
   Complete solution: $y = c_1 e^{-x} + c_2 e^{4x}$
   and we get: $y' = -c_1 e^{-x} + 4c_2 e^{4x}$

Thus $\begin{cases} c_1 + c_2 = 0 \\ -c_1 + 4c_2 = -5 \end{cases} \rightsquigarrow \begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix}$
Final result: $y = e^{-x} - e^{4x}$

(d) $y'' + 4y' + 29y = 0,\ y(0) = 0, y'(0) = 15$
$\lambda^2 + 4\lambda + 29 = 0,\ \lambda_{1,2} = -2 \pm 5i$
Complete solution: $y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$
and we get: $y' = e^{-2x}((5c_2 - 2c_1)\cos 5x + (-5c_1 - 2c_2)\sin 5x)$
Thus $\begin{cases} c_1 = 0 \\ 5c_2 - 2c_1 = 15 \end{cases} \rightsquigarrow \begin{matrix} c_1 = 0 \\ c_2 = 3 \end{matrix}$
Final result: $y = 3e^{-2x}\sin 5x$

(e) $y'' - 4y' + 13y = 0,\ y(0) = 0, y'(0) = 3$
$\lambda^2 - 4\lambda + 13 = 0,\ \lambda_{1,2} = 2 \pm 3i$
Complete solution: $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$
and we get: $y' = e^{2x}((2c_1 + 3c_2)\cos 3x + (2c_2 - 3c_1)\sin 3x)$
Thus $\begin{cases} c_1 = 0 \\ 2c_1 + 3c_2 = 3 \end{cases} \rightsquigarrow \begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}$
Final result: $y = e^{2x}\sin 3x$

3. Find a 2-order ODE, the solutions of which contain $1, e^x, 2e^x$.
   Note: $1 and e^x$ are two independent solutions.
   Thus, $y(x) = C_1 + C_2e^x$;
   So we can get:$\lambda_1 = 0, \lambda_2 = 1$.
   $\lambda^2 - \lambda = 0$
   Final result:$y'' - y' = 0$

2 Equations with non-constant coefficients

Find general solutions of the following equations. Show the details of your calculation.

1. $y'' - \frac{y'}{x} + \frac{y}{x^2} = \frac{2}{x}$
   Answer: $x(c_1 + c_2)\ln x + x\ln^2 x$

2. $x^2y'' - xy' + 4y = x \sin lnx$
   Answer: $x(c_1 \cos(\sqrt{3}\ln x) + c_2 \sin(\sqrt{3}\ln x)) + \frac{x}{2}\sin(\ln x)$

3. Prove that $y_1 = e^{x^2}$ and $y_2 = xe^{x^2}$ are solutions of the ODE $y'' - 4xy' + (4x^2 - 2)y = 0$, and write the general solution.
   Answer: $y = (C_1 + C_2x)e^{x^2}$

4. Solve the ODE $(2x-1)y'' - (2x+1)y' + 2y = 0$ in the general case (no initial value), knowing that $y_1(x) = e^x$ is a solution.
   Answer: $y = C_1(2x + 1) + C_2e^x$

5. Solve the ODE $x^2y'' - 2xy' + 2y = 0$ in the general case (no initial value), knowing that $y_1(x) = x$ is a solution.
   Answer:$y = C_1x + C_2x^2$

3 Nonhomogeneous ODEs

1. Solve the following ODEs:

   (a) $y'' + 3y' + 2y = 10e^{-3x}$
   $y(x) = C_1e^{-2x} + C_2e^{-x} + 5e^{-3x}$

   (b) $y'' + 5y' + 4y = x^2$
   $y(x) = C_1e^{-4x} + C_2e^{-x} + \frac{x^2}{4} - \frac{5x}{8} + \frac{21}{32}$

   (c) $y'' + 4y' + 4y = \cos x$
   $y(x) = C_1e^{-2x} + C_2e^{-2x} + \frac{4\sin x}{25} + \frac{3\cos x}{25}$

   (d) $y'' - 5y' + 6y = xe^{2x}$
   $\lambda^2 - 5\lambda + 6 - 0 \ \lambda_1 = 2, \lambda_2 = 3$
   Complete solution: $Y = c_1e^{2x} + c_2e^{3x}$
   Particular solution: $y^* = x(-\frac{1}{2}x - 1)e^{2x}$
   Final result: $y = c_1e^{2x} + c_2e^{3x} - \frac{1}{2}(x^2 + 2x)e^{2x}$

   (e) $y'' + y = x\cos 2x$
   $y = -\frac{1}{3}x\cos 2x + \frac{4}{9}\sin 2x$

   (f) $y'' - y = e^x\cos 2x$
   $y = c_1e^x + c_2e^{-x} + \frac{1}{8}e^x(\sin 2x - \cos 2x)$

2. Solve the ODE with the given initial values:

   (a) $y'' + 4y = x^2, y(0) = 1, y'(0) = 0$
   $y(x) = \frac{1}{8}(2x^2 + 9\cos(2x) - 1)$

   (b) $y'' - 2y' = 6e^{2x} - 4e^{-2x}, y(0) = -1, y'(0) = 6$
   $y(x) = e^{2x}(3x + 1) - \frac{e^{-2x}}{2} - \frac{3}{2}$

   (c) $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, y(0) = 2.78, y'(0) = -0.43$
   $y = 3.1e^{-\frac{x}{2}} + \sin x + 0.12x - 0.32$

4 High order ODE

1. $y^{(4)} - 2y^{(3)} + 5y'' = 0$
   $\lambda^4 - 2\lambda^3 + 5\lambda^2 = 0$
   $\lambda^2(\lambda^2 - 2\lambda + 5) = 0$
   $\lambda_1 = \lambda_2 = 0, \lambda_{3,4} = 1 \pm 2i$
   Complete solution: $y = c_1 + c_2x + e^x(c_3\cos 2x + c_4\sin 2x)$

2. $y^{(4)} + y = 0$
   $\lambda^4 + \beta^4 = 0$
   $\lambda^4 + \beta^4 = \lambda^4 + 2\lambda^2\beta^2 + \beta^4 - 2\lambda^2\beta^2 = (\lambda^2 + \beta^2)^2 - 2\lambda^2\beta^2 = (\lambda^2 - \sqrt{2}\beta\lambda + \beta^2)(\lambda^2 + \sqrt{2}\beta\lambda + \beta^2)$
   Thus, $\lambda_{1,2} = \frac{\beta}{\sqrt{2}}(1 \pm i)$, $\lambda_{3,4} = -\frac{\beta}{\sqrt{2}}(1 \pm i)$
   Complete solution: $y = e^{\frac{\beta}{\sqrt{2}}x}(c_1\cos\frac{\beta}{\sqrt{2}}x + c_2\sin\frac{\beta}{\sqrt{2}}x) + e^{-\frac{\beta}{\sqrt{2}}x}(c_3\cos\frac{\beta}{\sqrt{2}}x + c_4\sin\frac{\beta}{\sqrt{2}}x)$

3. $y''' = e^{2x} - cosx$
   Note: Intergrate 3 times
   Complete solution:$y = \frac{1}{8}e^{2x} + sinx + C_1x^2 + C_2x + C_3(C_1 = \frac{C_2}{2})$

4. $y''' + 3y'' + 3y' + y = 0$
   Note: $(\lambda + 1)^3 = 0$
   $y(x) = C_3 e^{-x} x^2 + C_2 e^{-x} x + C_1 e^{-x}$