Problem Bank 7: Partial Differential Equation

Kreyszig Section	Topics
12.1	**Basic Concepts.**
12.2-3	**Wave Equation. Separating Variables.**
	Use of Fourier Series.
12.6	**Heat equation.**

1 Basic Concepts.

1. **Classification of PDEs**
   Classify the following equations in terms of its order, linearity and homogeneity (if the equation is linear).

   (a) $u_t - u_{xx} + 1 = 0$

   Solution: Second order, linear and non-homogeneous.

   (b) $u_t - u_{xx} + xu = 0$

   Solution: Second order, linear and homogeneous.

   (c) $u_t - u_{xxt} + uu_x = 0$

   Solution: Order 3 and non-linear.

   (d) $u_{tt} - u_{xx} + x^2 = 0$

   Solution: Second order, linear and non-homogeneous.

   (e) $\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_y}{\sqrt{1+u_y^2}} = 0$

   Solution: First order and non-linear.

   (f) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

   Solution: Order 4 and non-linear.

2. Verify that for all pairs of differential functions f and g of one variable, $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$.

Solution: First, compute u_x, u_y and u_{xy}:

$$u_x = g(y)f'(x)$$
$$u_y = f(x)g'(y)$$
$$u_{xy} = f'(x)g'(y)$$

Substituting into the PDE, we have

$$uu_{xy} = f(x)g(y)f'(x)g'(y) = u_x u_y$$

Hence, $u(x,y) = f(x)g(y)$ is a solution of the PDE.

3. **Boundary value problem**
The Poisson's Equation is the non-homogeneous version of Laplace's Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y) \tag{1}$$

Assume that $\rho(x,y) = 1$.

(a) Find the condition under which $u(x,y) = C_1 x^2 + C_2 y^2$ is a solution to the Poisson's Equation above.

Solution: First, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2C_1 + 2C_2$. Since $\rho(x,y) = 1$, we have $\boxed{C_1 + C_2 = \frac{1}{2}}$.

(b) Suppose we also have the boundary condition

$$u(0,y) = \frac{y^2}{4}$$

Determine C_1 and C_2.

Solution: $u(0,y) = C_1(0)^2 + C_2 y^2 = \frac{y^2}{4} \longrightarrow \boxed{C_2 = \frac{1}{4}}$. Hence, $\boxed{C_1 = \frac{1}{2} - C_2 = \frac{1}{4}}$.

(c) Find another solution that satisfies both the Poisson's Equation and the boundary condition.

Solution: Since $f(x,y) = \frac{1}{4}(x^2 + y^2)$ satisfies the Poisson's Equation, if we find any harmonic function $g(x,y)$ such that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$, the sum of f and g will continue to satisfy the Poisson's Equation. The boundary condition will be satisfied if $g(0, y) = 0$. One possible choice is therefore $g(x, y) = x$, which gives

$$u(x, y) = x + \frac{1}{4}(x^2 + y^2)$$

4. **Initial–boundary value problem**
Suppose a metal rod of length L has an initial temperature of $\sin\left(\frac{\pi}{L}x\right)$ and the temperatures at its left and right ends are both fixed at 0 degree Celsius. What would be the initial–boundary value problem that describes this scenario?

Solution:

$$\begin{array}{ll} \text{PDE} & u_t = c^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}, \quad 0 < t < \infty \\ \text{ICs} & \begin{cases} u(x,0) = \sin\left(\frac{\pi}{L}x\right) \\ u_t(x,0) = 0 \end{cases}, \quad 0 \le x \le L \end{array}$$

5. **Principle of Superposition**

(a) Let $\mathcal{D}$ be a unit disk centered at (0,0), i.e., $\mathcal{D}$ includes all the points falling inside the unit circle $x^2 + y^2 = 1$. Suppose $f_1(x, y) = x^2 - y^2$ is a solution to the following boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = 2x^2 - 1, & \text{for } x^2 + y^2 = 1 \end{cases}$$

while $f_2(x, y) = x$ is a solution to the boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = x, & \text{for } x^2 + y^2 = 1 \end{cases}$$

Solve the following boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = \frac{3}{4}(2x^2 - 1) + \frac{3}{5}x, & \text{for } x^2 + y^2 = 1 \end{cases}$$

Solution: By the principle of superposition,

$$u(x, y) = \frac{3}{4}(x^2 - y^2) + \frac{3}{5}x$$

will satisfy both the Laplace's Equation and the boundary condition.

(b) Suppose $f_1(x,t) = \sin(\pi(x - ct))$ and $f_2(x,t) = \sin(3\pi(x + ct))$ both satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and the boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = 0 \qquad \text{for all } t$$

In addition, $u_1(x,t)$ and $u_2(x,t)$ satisfy the initial condition $u(x,0) = \sin(\pi x)$ and $u(x,0) = \sin(3\pi x)$, respectively. Find a solution that satisfies the initial condition $u(x,0) = \sin(\pi x) - \sin(3\pi x)$.

Solution: $u(x,t) = f_1(x,t) - f_2(x,t) = \sin(\pi(x - ct)) - \sin(3\pi(x + ct))$

6. **Steady state solution**
The heat equation for a rod with a constant internal heat source is described by the following PDE:

$$u_t = c^2 u_{xx} + 1, \quad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0,t) = 0$ and $u(1,t) = 1$. Will the temperature $u(x,t)$ converge to a constant temperature $U(x)$ independent of time? If so, what is this constant temperature?
HINT: Set $u_t = 0$.

Solution: Set $u_t = 0$, we have $u_{xx} = -\frac{1}{c^2}$. Integrating, we have

$$u(x) = -\frac{1}{c^2}\frac{x^2}{2} + Ax + B$$

Putting in the boundary conditions, we can find that $B = 0$ and $A = 1 + \frac{1}{2c^2}$. Hence the steady state temperature is given by

$$U(x) = -\frac{1}{2c^2}(x^2 - x) + x$$

7. Solve the following PDEs

(a) $u_{yy} = (\cosh x)\, yu$;

(b) $u_y = 2xyu$;

(c) $u_{xx} = 0, u_{yy} = 0$,

where $u = u(x,y)$.

Solution:

(a) We need to solve the partial differential equation

$$u_y = (\cosh x)yu.$$

Since no derivative of x occurs, the partial differential equation becomes

$$u' - (\cosh x)yu = 0 \Longrightarrow \frac{du}{u} = (\cosh x)ydy.$$

After integrating, we get

$$\ln u = \frac{1}{2}(\cosh x)y^2 + \ln A,$$

where we have

$$u = Ae^{\frac{1}{2}(\cosh x)y^2} \Longrightarrow u(x,y) = A(x)e^{\frac{1}{2}(\cosh x)y^2}$$

(b) We need to solve the partial differential equation

$$u_y = 2xyu.$$

Since no derivative of x occurs, the partial differential equation becomes

$$u' = 2xyu \Rightarrow \frac{u'}{u} = 2xy \Rightarrow \frac{du}{u} = 2xydy \Rightarrow \ln u = xy^2 + \ln A \Rightarrow u = Ae^{xy^2}.$$

Thus, we have $u(x,y) = A(x)e^{xy^2}$.

(c) Integrating the first PDE and the second PDE gives

$$u = c_1(y)x + c_2(y) \text{ and } u = c_3(x)y + c_4(x),$$

respectively. Equating these two functions gives

$$u = axy + bx + cy + k.$$

Alternatively, $u_{xx} = 0$ gives $u = c_1(y)x + c_2(y)$. Then from $u_{yy} = 0$, we get $u_{yy} = c_1''x + c_2'' = 0$, hence $c_1'' = 0, c_2'' = 0$, and by integration

$$c_1 = \alpha y + \beta, \quad c_2 = \gamma y + \delta$$

and by substitution in the previous expression

$$u = c_1x + c_2 = \alpha xy + \beta x + \gamma y + \delta.$$

2 Wave Equation. Separating Variables. Use of Fourier Series.

1. **D'Alembert Solution of the Wave Equation**.
   Using the D'Alembert solution, find the solution to the following initial-value problem

   (a)

$$\text{PDE} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$
$$\text{ICs} \quad \begin{cases} u(x,0) = e^{-x^2} \\ u_t(x,0) = 0 \end{cases}, \quad -\infty < x < \infty$$

   (b)

$$\text{PDE} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$
$$\text{ICs} \quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = xe^{-x^2} \end{cases}, \quad -\infty < x < \infty$$

Solution: Applying the D'Alembert solutions directly, we have

(a) $u(x,t) = \frac{1}{2}(e^{-(x-ct)^2} + e^{-(x+ct)^2})$

(b) $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} ue^{-u^2}du = \frac{-1}{4c}\int_{x-ct}^{x+ct} e^{-u^2}d(-u^2) = \frac{1}{4c}\{e^{-(x-ct)^2} - e^{-(x+ct)^2}\}$

2. **Wave equation and standing waves**.
   Find $u(x,t)$ for the string of length $L = 1$ and $c^2 = 1$ when the string is fixed at the two ends (i.e., $u(0,0) = u(L,0) = 0$), the initial velocity is zero (i.e., $u_t(x,0) = 0$) and the initial shape (i.e., $u(x,0) = 0$) of the string is given by the specified function.

   (a) $u(x,0) = kx(1-x^2)$

Solution:
It's given that $u_{tt} = u_{xx}$ and $L = 1$ with condition that

$$u_t(x,0) = 0, \text{ and } u(x,0) = f(x) = kx(1-x^2).$$

The solution of $u_{tt} = u_{xx}$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x,$$

where $B_n = 2\int_0^1 f(x)\sin n\pi x dx = 2k\int_0^1 x(1-x^2)\sin n\pi x dx$.

$\int_0^1 x(1-x^2)\sin n\pi x dx$ can be obtained by using integration by parts

$$\int_0^1 x(1-x^2)\sin n\pi x dx = \frac{-6\pi n \cos n\pi}{\pi^4 n^4} = \frac{-6}{\pi^3 n^3}(-1)^n.$$

Thus we have $B_n = -\frac{12k}{\pi^3 n^3}(-1)^n$. Therefore we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x = \sum_{n=1}^{\infty} -\frac{12k}{\pi^3 n^3}(-1)^n \cos n\pi t \sin n\pi x$$

(b) $u(x,0) = kx(1-x)$

Solution:
It is given that $u_{tt} = u_{xx}$ and $L = 1$ with condition that

$$u_t(x,0) = 0, \text{ and } u(x,0) = f(x) = kx(1-x). \tag{2}$$

For $L = 1$, the solution of $u_{tt} = u_{xx}$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x, \tag{3}$$

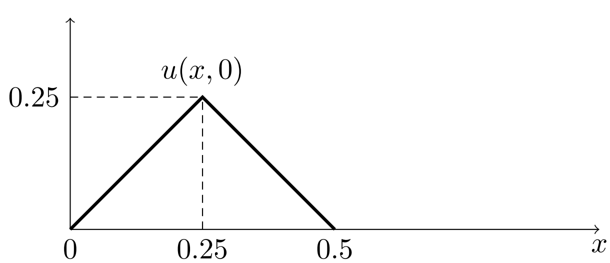
where $B_n = 2\int_0^1 kx(1-x)\sin n\pi x dx = 2k\int_0^1 x(1-x)\sin n\pi x dx$. After integration, we have

$$B_n = 2k\left[\left(-\frac{1}{n\pi}\right)(x(1-x)\cos n\pi x)_0^1 + \frac{1}{n\pi}\left\{\frac{1}{n\pi}((1-2x)\sin n\pi x)_0^1 - \frac{2}{n^2\pi^2}(\cos n\pi x)_0^1\right\}\right]$$
$$= 2k\left[\frac{2}{n^3\pi^3}(1-(-1)^n)\right] = \frac{4k(1-(-1)^n)}{n^3\pi^3}.$$

So we have

$$u(x,t) = \frac{4k}{\pi^3}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}\cos n\pi t \sin n\pi x. \tag{4}$$

(c) given by the following graph:



Solution:
It is given that $u_{tt} = u_{xx}$ and $L = 1$ with conditions that

$$u_t(x,0) = 0 \text{ and } u(x,0) = f(x) = \begin{cases} x & \text{if } 0 < x < 1/4 \\ \frac{1}{2} - x & \text{if } 1/4 < x < 1/2 \\ 0 & \text{if } 1/2 < x < 1 \end{cases}.$$

The solution of $u_{tt} = u_{xx}$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x,$$

where $B_n = 2\int_0^1 f(x)\sin n\pi x dx = 2\left[\int_0^{1/4} x \sin n\pi x dx + \int_{1/4}^{1/2}\left(\frac{1}{2}-x\right)\sin n\pi x dx\right]$.
After integration, we have

$$B_n = 2\left[\left(-\frac{1}{n\pi}\right)(x\cos n\pi x)_0^{\frac{1}{4}} + \frac{1}{n^2\pi^2}(\sin n\pi x)_0^{\frac{1}{4}} - \frac{1}{n\pi}\left\{\left(\frac{1}{2}-x\right)\cos n\pi x\right\}_{\frac{1}{4}}^{\frac{1}{2}} - \frac{1}{n\pi}(\sin n\pi x)_{\frac{1}{4}}^{\frac{1}{2}}\right]$$
$$= \frac{2}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right).$$

Therefore, we have the solution as

$$u(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right)\cos n\pi t \sin n\pi x.$$

Hint: If the initial velocity of the string is zero, the solution of the problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

3. **Method of separating variables**
   By the method of separating variables, write down the general solution for the equation

$$tu_t + u_{xx} + 2u = 0$$

under the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

You may proceed as follows:

(a) Search for separable solutions of the form $u(x,t) = f_1(t)f_2(x)$, to obtain two equations:

$$\begin{cases} \frac{tf_1'}{f_1} = -\lambda \\ \frac{f_2''+2f_2}{f_2} = -\lambda \end{cases}$$

(b) When solving for $f_2(x)$, examine the following cases: $\lambda+2>0$, $\lambda+2=0$ and $\lambda+2<0$. You should find that only in the first case can you find solutions that satisfy the boundary conditions.

(c) Write down the general solution.

Solution:

- Let $u(x,t) = f_1(t)f_2(x)$, then we have

$$tf_2f_1' = f_1f_2'' + 2f_1f_2$$

- Separating variables and let the term on the two sides be $-\lambda$:

$$\frac{tf_1'}{f_1} = \frac{f_2'' + 2f_2}{f_2} = -\lambda$$

- The equation associated with t is

$$\begin{aligned} tf_1' = -\lambda f_1 \longrightarrow \frac{df_1}{f_1} = -\lambda\frac{dt}{t} \\ f_1(t) = Ae^{-\lambda \log t} \end{aligned}$$

- The equation associated with x is

$$\frac{f_2'' + 2f_2}{f_2} = -\lambda \longrightarrow f_2'' + (\lambda+2)f_2 = 0$$

- For $\lambda + 2 > 0$, let $2 + \lambda = \beta^2$, we have $f_2'' + \beta^2 f_2 = 0$. So,

$$f_2(x) = C\cos\beta x + D\sin\beta x$$

  Apply boundary conditions:

$$\begin{cases} f_2(0) = 0 \longrightarrow C = 0 \\ f_2(\pi) = 0 \longrightarrow D\sin\beta\pi = 0 \end{cases}$$

  which gives $\beta\pi = n\pi \longrightarrow \beta = n(n = 1, 2, 3, \ldots)$

- For $\lambda + 2 = 0$, the equation becomes $f_2'' = 0$, and $f_2(x) = Ax + B$. The boundary solutions are satisfied only when $A = B = 0$, thus the solution of this form is not useful.

- For $\lambda + 2 < 0$, the equation can be re-written as $f_2'' - \gamma^2 f_2 = 0$, where γ is positive and $-\gamma^2 = \lambda + 2$. The solution in this case is $f_2 = Ae^{\gamma x} + Be^{-\gamma x}$. Again, the boundary solutions are satisfied only when $A = B = 0$, thus the solution of this form is not useful.

- The general solution is thus

$$u(x,t) = \sum_n (A_n \sin nx)e^{-\lambda \log t}$$

3 Heat Equation.

1. Show that for a completely insulated bar described by the following boundary conditions:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

and the following initial condition:

$$u(x,0) = f(x)$$

separation of variables gives the following solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(\frac{cn\pi}{L})^2 t}$$

where A_0 and A_n are given by

$$A_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 1, 2, \cdots$$

Solution: Since left hand side depends only on t and the right hand side only on x. So that both sides must be constant say k. For $k = 0$ and $k > 0$, we get only trivial solution i.e, $u = 0$, so we do not consider them.

Now for negative $k = ip^2, (p \neq 0)$, we have:

$$\frac{T'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -p^2$$

$$X''(x) + p^2X(x) = 0 \quad (1) \qquad \text{and} \qquad T'(t) + c^2p^2T(t) = 0 \quad (2)$$

Now solution of the equation (1) is given by:

$$X(x) = A\cos px + B\sin bx$$

$X'(0) = 0$, so $Bp = 0$, and $B = 0$. Also, $X'(L) = 0 = -Ap\sin pl \Longrightarrow \sin pl = 0 \Longrightarrow p = \frac{n\pi}{L}, n = 1, 2, 3, \cdots$ Setting $A = 1$, we get the solution

$$X_n(x) = \cos(\frac{x\pi}{L})$$

Now, we solve equation (2) for $p = \frac{x\pi}{L}$:

$$T'(t) + c^2\frac{n^2\pi^2}{L^2}T(t) = 0$$
$$T'(t) + \lambda^2n^2T(t) = 0$$

where $\lambda_n = \frac{cn\pi}{L}$.

So its general solution is given by:

$$T_n = A_n e^{-\lambda n^2 t}, \qquad n = 0, 1, 2, \cdots$$

where A_n is a constant.

Hence, the functions

$$u_n(n.t) = X_n(x)T_n(t) = A_n \cos(\frac{nx\pi}{L})e^{-n^2 t\lambda}$$

are the solutions of heat equation.

These are the eigenfunctions of the problem corresponding to the eigenvalues

$$\lambda_n = \frac{cn\pi}{L}$$

By the principle of superposition:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{nx\pi}{L})e^{-n^2 t\lambda}$$

We have $\lambda_n = \frac{nc\pi}{L}$.

Now,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n cos(\frac{nx\pi}{L})$$

where A_n is given by

$$A_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{nx\pi}{L})e^{\frac{-c^2 n^2 \pi^2 t}{L^2}}$$

Thus, the general solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{nx\pi}{L})e^{\frac{-c^2 n^2 \pi^2 t}{L^2}}$$

2. Find the temperature in problem 1 above with $L = \pi$, $c = 1$, and $f(x) = 0.5\cos 4x$.

Solution:
We need to solve the one-demensional equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions:

$$\begin{cases} u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \end{cases}$$

and the initial condition

$$u(x,0) = f(x) = 0.5\cos 4x$$

We know that the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos(nx)e^{-n^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)e^{-n^2}$$

Now,

$$u(x,0) = f(x) = 0.5\cos 4x = \sum_{n=1}^{\infty} A_n \cos(nx)$$

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} 0.5\cos(4x)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(4x)\cos(nx)dx \\ &= \begin{cases} \frac{1}{2} & \text{for} \quad n = 4 \\ 0 & \text{for} \quad n \neq 4 \end{cases} \end{aligned}$$

That is, the only term that does not vanish is for $n = 4$. Thus, the solution is given by:

$$u(x,t) = \frac{1}{2}\cos(4x)e^{-16t}$$