Problem Bank 7: Partial Differential Equation

Kreyszig Section	Topics
12.1	**Basic Concepts.**
12.2-3	**Wave Equation. Separating Variables. Use of Fourier Series.**
12.6	**Heat equation.**

1 Basic Concepts.

1. **Classification of PDEs**
   Classify the following equations in terms of its order, linearity and homogeneity (if the equation is linear).

   (a) $u_t - u_{xx} + 1 = 0$

   (b) $u_t - u_{xx} + xu = 0$

   (c) $u_t - u_{xxt} + uu_x = 0$

   (d) $u_{tt} - u_{xx} + x^2 = 0$

   (e) $\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_y}{\sqrt{1+u_y^2}} = 0$

   (f) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

2. Verify that for all pairs of differential functions f and g of one variable, $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$.

3. **Boundary value problem**
   The Poisson's Equation is the non-homogeneous version of Laplace's Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x,y) \qquad (1)$$

   Assume that $\rho(x,y) = 1$.

   (a) Find the condition under which $u(x,y) = C_1 x^2 + C_2 y^2$ is a solution to the Poisson's Equation above.

   (b) Suppose we also have the boundary condition

$$u(0,y) = \frac{y^2}{4}$$

   Determine C_1 and C_2.

(c) Find another solution that satisfies both the Poisson's Equation and the boundary condition.

4. **Initial–boundary value problem**
Suppose a metal rod of length L has an initial temperature of $\sin\left(\frac{\pi}{L}x\right)$ and the temperatures at its left and right ends are both fixed at 0 degree Celsius. What would be the initial–boundary value problem that describes this scenario?

5. **Principle of Superposition**

(a) Let $\mathcal{D}$ be a unit disk centered at (0,0), i.e., $\mathcal{D}$ includes all the points falling inside the unit circle $x^2 + y^2 = 1$. Suppose $f_1(x, y) = x^2 - y^2$ is a solution to the following boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = 2x^2 - 1, & \text{for } x^2 + y^2 = 1 \end{cases}$$

while $f_2(x, y) = x$ is a solution to the boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = x, & \text{for } x^2 + y^2 = 1 \end{cases}$$

Solve the following boundary-value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in \mathcal{D} \\ u(x, y) = \frac{3}{4}(2x^2 - 1) + \frac{3}{5}x, & \text{for } x^2 + y^2 = 1 \end{cases}$$

(b) Suppose $f_1(x, t) = \sin(\pi(x - ct))$ and $f_2(x, t) = \sin(3\pi(x + ct))$ both satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and the boundary conditions:

$$u(0, t) = 0, \quad u(1, t) = 0 \qquad \text{for all } t$$

In addition, $u_1(x, t)$ and $u_2(x, t)$ satisfy the initial condition $u(x, 0) = \sin(\pi x)$ and $u(x, 0) = \sin(3\pi x)$, respectively. Find a solution that satisfies the initial condition $u(x, 0) = \sin(\pi x) - \sin(3\pi x)$.

6. **Steady state solution**
The heat equation for a rod with a constant internal heat source is described by the following PDE:

$$u_t = c^2 u_{xx} + 1, \quad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0, t) = 0$ and $u(1, t) = 1$. Will the temperature $u(x, t)$ converge to a constant temperature $U(x)$ independent of time? If so, what is this constant temperature?
HINT: Set $u_t = 0$.

7. Solve the following PDEs

   (a) $u_{yy} = (\cosh x)\, yu$;

   (b) $u_y = 2xyu$;

   (c) $u_{xx} = 0, u_{yy} = 0$,

   where $u = u(x, y)$.

2 Wave Equation. Separating Variables. Use of Fourier Series.

1. **D'Alembert Solution of the Wave Equation**.
   Using the D'Alembert solution, find the solution to the following initial-value problem

   (a)

$$\text{PDE} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$
$$\text{ICs} \quad \begin{cases} u(x,0) = e^{-x^2} \\ u_t(x,0) = 0 \end{cases}, \quad -\infty < x < \infty$$

   (b)

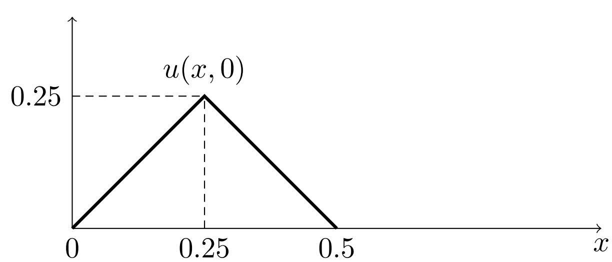
$$\text{PDE} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$
$$\text{ICs} \quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = xe^{-x^2} \end{cases}, \quad -\infty < x < \infty$$

2. **Wave equation and standing waves**.
   Find $u(x, t)$ for the string of length $L = 1$ and $c^2 = 1$ when the string is fixed at the two ends (i.e., $u(0, 0) = u(L, 0) = 0$), the initial velocity is zero (i.e., $u_t(x, 0) = 0$) and the initial shape (i.e., $u(x, 0) = 0$) of the string is given by the specified function.

   (a) $u(x, 0) = kx(1 - x^2)$

   (b) $u(x, 0) = kx(1 - x)$

   (c) given by the following graph:



Hint: If the initial velocity of the string is zero, the solution of the problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right)$$

3. **Method of separating variables**
   By the method of separating variables, write down the general solution for the equation

$$tu_t + u_{xx} + 2u = 0$$

   under the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

   You may proceed as follows:

   (a) Search for separable solutions of the form $u(x,t) = f_1(t)f_2(x)$, to obtain two equations:

$$\begin{cases} \frac{tf_1'}{f_1} = -\lambda \\ \frac{f_2''+2f_2}{f_2} = -\lambda \end{cases}$$

   (b) When solving for $f_2(x)$, examine the following cases: $\lambda+2>0$, $\lambda+2=0$ and $\lambda+2<0$. You should find that only in the first case can you find solutions that satisfy the boundary conditions.

   (c) Write down the general solution.

3 Heat Equation.

1. Show that for a completely insulated bar described by the following boundary conditions:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

and the following initial condition:

$$u(x,0) = f(x)$$

separation of variables gives the following solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

where A_0 and A_n are given by

$$A_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 1, 2, \cdots$$

2. Find the temperature in problem 1 above with $L = \pi$, $c = 1$, and $f(x) = 0.5 \cos 4x$.