Solution 6: Fourier Analysis

1 Fourier Series

1. (a) 2π, 2π, π, π, 2, 2, 1, 1;
   (b) $2\pi/n$, $2\pi/n$, k, k, k/n, k/n;
   (c) $f(x+p) = f(x)$ implies $f(ax+p) = f(a[x+(p/a)]) = f(ax)$ or $g[x+(p/a)] = g(x)$, where $g(x) = f(ax)$. Thus $g(x)$ has the period p/a. This proved the first statement. The other statement follows by setting $a = 1/b$.

2. (a) N/A;
   (b) N/A;
   (c) N/A;

3. (a) $\frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x - \cdots$
   (b) $\frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + 2\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$
   (c) $\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x - + \cdots\right)$
   (d) $\frac{2}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right) + \left(\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x - + \cdots\right)$

2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

1. (a) Neither, even, odd, odd, neither;
   (b) Even, even, neither, odd, even;
   (c) (*) Odd for sums and for products of an odd number $2k+1$ for factors, $f(-x) = f_1(-x)\cdots f_{2k+1}(-x) = (-1)^{2k+1} f_1(x)\cdots f_{2k+1}(x) = -f(x)$. Even for products of an even number of factors;
   (d) (*) Odd. This is important in connection with the integrand in the Euler formulas for the Fourier coefficients. It implies the simplification of the Fourier series of an odd function to a Fourier sine series and of the Fourier series of an even function to a Fourier cosine series.

2. (a) Even, $L = 1$. $\frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$
   (b) Even, $L = 1$. $\frac{1}{3} - \frac{4}{\pi^2}\left(\cos\pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$
   (c) Even, $L = 1/2$, full-wave rectification of a cosine current. $\frac{2}{\pi} + \frac{4}{\pi}(\frac{1}{1\cdot 3}\cos 2\pi x - \frac{1}{3\cdot 5}\cos 4\pi x + \frac{1}{5\cdot 7}\cos 6\pi x - + \cdots)$
   (d) Odd, $L = 1$. $\frac{2}{\pi^3}((\pi^2 - 4)\sin\pi x + \frac{1}{27}(9\pi^2 - 4)\sin 3\pi x + \frac{1}{125}(25\pi^2 - 4)\sin 5\pi x + \cdots) - \frac{1}{\pi}(\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \cdots)$

(e) Even, $L = 1$. $\frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$

(f) (*)Set $x = 1$. Then $1 = f(1) = \frac{1}{3} + \frac{4}{\pi^2}(1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots)$. Hence, the results.

3. (a) $L = 4$, **(a)** $\frac{1}{2} - \frac{2}{\pi}\left(\cos \frac{\pi x}{4} - \frac{1}{3}\cos \frac{3\pi x}{4} + \frac{1}{5}\cos \frac{5\pi x}{4} - + \cdots\right)$,
(b) $\frac{2}{\pi}\left(\sin \frac{\pi x}{4} + \frac{1}{3}\sin \frac{3\pi x}{4} + \frac{1}{5}\sin \frac{5\pi x}{4} + \cdots\right) - \frac{2}{\pi}\left(\sin \frac{\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} + \frac{1}{5}\sin \frac{5\pi x}{2} + \cdots\right)$

(b) $L = \pi$, **(a)** $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$,
(b) $2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots\right)$

(c) $L = \pi$, **(a)** $\frac{3\pi}{8} - \frac{2}{\pi}(\cos x + \frac{1}{2}\cos 2x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \frac{1}{18}\cos 6x + \frac{1}{49}\cos 7x + \cdots)$
(b) $(1 + \frac{2}{\pi})\sin x - \frac{1}{2}\sin 2x + (\frac{1}{3} - \frac{2}{9\pi})\sin 3x - \frac{1}{4}\sin 4x + (\frac{1}{5} + \frac{2}{25\pi})\sin 5x - \frac{1}{6}\sin 6x + (\frac{1}{7} - \frac{2}{49\pi}\sin 7x) + \cdots$

(d) **(a)** $\frac{L}{2} - \frac{4L}{\pi^2}(\cos \frac{\pi x}{L} + \frac{1}{9}\cos \frac{3\pi x}{L} + \frac{1}{25}\cos \frac{5\pi x}{L} + \cdots)$
(b) $\frac{2L}{\pi}(\sin \frac{\pi x}{L} - \frac{1}{2}\sin \frac{2\pi x}{L} + \frac{1}{3}\sin \frac{3\pi x}{L} - \frac{1}{4}\sin \frac{4\pi x}{L} + - \cdots)$

3 Forced Oscillations

1. (a) $y = C_1 \sin \omega t + C_2 \cos \omega t + \frac{\sin \alpha t}{\omega^2 - \alpha^2} + \frac{\sin \beta t}{\omega^2 - \beta^2}$;

(b) $y = C_1 \sin \omega t + C_2 \cos \omega t + a(\omega)\cos t$, $a(\omega) = \frac{1}{\omega^2 - 1} = -1.19, -2.78, -5.26, 4.76, 2, 27, 0.0417$ resonating;

(c) $y = C_1 \cos \omega t + C_2 \sin \omega t + \frac{1}{2\omega^2} - \frac{1}{1\cdot 3(\omega^2 - 4)}\cos 2t - \frac{1}{3\cdot 5(\omega^2 - 16)}\cos 4t - \frac{1}{5\cdot 7(\omega^2 - 36)}\cos 6t - \cdots$.

(d) $y = C_1 \cos \omega t + C_2 \sin \omega t + \frac{4}{\pi}(\frac{\sin t}{\omega^2 - 1} + \frac{1}{3}\frac{\sin 3t}{\omega^2 - 9} + \frac{1}{5}\frac{\sin 5t}{\omega^2 - 25} + \cdots)$

2. (a) The Fourier series is a Fourier sine series, as given and derived in Example 1 of ($k = 1$) Sec. 11.1 with coefficients $b_n = 4/(n\pi)$, (n odd). Hence the ODE must be solved with the right side $r_n(t) = (4/n\pi)\sin nt$, (n odd). The steady-state solution of this ODE is $y = \sum_{n=1, n \text{ odd}}^{\infty}(A_n \cos nt + B_n \sin nt)$, where $A_n = -\frac{4}{n\pi}\frac{cn}{(1-n^2)^2 + c^2n^2}$, $B_n = \frac{4}{n\pi}\frac{1-n^2}{(1-n^2)^2 + c^2n^2}$.

(b) For the right side we have the Fourier sine series $\frac{4}{\pi}(\sin t - \frac{1}{9}\sin 3t + \frac{1}{25}\sin 5t - + \cdots)$ with the coefficients $b_n = 4/(n^2\pi)$ if $n = 1, 5, 9, \cdots$ and $b_n = -4/(n^2\pi)$ if $n = 3, 7, 11, \cdots$. Substitution of this series into the ODE gives $y = A_1 \cos t + B_1 \sin t + A_3 \cos 3t + B_3 \sin 3t + \cdots$ with coefficients $A_n = -ncb_n/D_n$, $B_n = (1 - n^2)b_n/D_n$, $D_n = (1 - n^2)^2 + n^2c^2$. The damping constant c appears in the cosine terms, causing a phase shift, which is zero if $c = 0$. Also, c increases D_n, hence it decreases the amplitudes, which is physically understandable.

3. The ODE in Problems 3(a) and 3(b) is the same, except for the changing right sides, whose Fourier series we use term-by-term, as in the text. The solution of the ODE is of the general form

$$I = A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$$

with coefficients obtained by substitution

$$A_n = -\frac{n^2 - 10}{D_n}a_n, B_n = \frac{10n}{D_n}a_n, D_n = (n^2 - 10)^2 + 100n^2,$$

in particular, $A_0 = a_0/10$, the ODE being

$$I'' + 10I' + 10I = a_n \cos nt.$$

For 3(b), we have the Fourier series

$$100 + 100\pi - \frac{800}{\pi}(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \cdots).$$

Hence $A_0 = 10 + 10\pi$ and all the other A_n and B_n with n even are zero. The formula for the a_n is $-800/(\pi n^2)$ where n is odd. Numerically evaluating the terms, we obtain the solution (the current in the RLC-circuit).

$$I = 41.416 - 12.662\cos t - 14.069\sin t - 0.031\cos 3t - 0.942\sin 3t + 0.056\cos 5t - 0.187\sin 5t + \cdots$$