Problem Bank 6: Fourier Analysis

Kreyszig Section No.s	Topics	Slide No.s
11.1	**Fourier Series**	**N/A**
11.2	**Arbitrary Period. Even and Odd Functions. Half-Range Expansions**	**N/A**
11.3	**Forced Oscillations**	**N/A**

1 Fourier Series

1. The fundamental period is the smallest positive period. Find it for

   (a) $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

   (b) $\cos nx$, $\sin nx$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi nx}{k}$, $\sin \frac{2\pi nx}{k}$;

   (c) If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b), b \neq 0$ are periodic functions of x of periods p/a and bp, respectively. Give examples;

2. Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows:

   (a) $f(x) = |x|$;

   (b) $f(x) = \begin{cases} x, & \text{if } -\pi < x < 0 \\ \pi - x, & \text{if } 0 < x < \pi \end{cases}$;

   (c) $f(x) = \begin{cases} -\cos^2 x, & \text{if } -\pi < x < 0 \\ \cos^2 x, & \text{if } 0 < x < \pi \end{cases}$;

3. Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work.

   (a) $f(x)$ in Problem 2(a)

   (b) $f(x)$ in Problem 2(b)

   (c) $f(x) = x^2 \quad (-\pi < x < \pi)$;

   (d) $f(x) = \begin{cases} 0, & \text{if } -\pi < x < -\pi/2 \\ x, & \text{if } -\pi/2 \leq x < \pi/2 \\ 0, & \text{if } \pi/2 \leq x < \pi \end{cases}$

2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

1. Are the following functions even or odd or neither even nor odd?
   (a) e^x, $e^{-|x|}$. $x^3 \cos nx$, $x^2 \tan \pi x$, $\sinh x - \cosh x$;
   (b) $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2+1)$, $x \cot x$;
   (c) (*) Sum and product of odd functions;
   (d) (*) Product of an odd times an even function.

2. Is the given function even or odd or neither even nor odd? Find its Fourier series(period $p = 2L$). Show details of your work.
   (a) $f(x) = \begin{cases} -x, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$;
   (b) $f(x) = x^2 \quad (-1 < x < 1),\ \ p = 2$;
   (c) $f(x) = \cos \pi x \quad (-1/2 < x < 1/2),\ \ p = 1$;
   (d) $f(x) = x|x| \quad (-1 < x < 1),\ \ p = 2$;
   (e) $f(x) = \begin{cases} x+1, & \text{if } -1 < x < 0 \\ -x+1, & \text{if } 0 < x < 1 \end{cases}$
   (f) (*)Using Problem 2(b), show that $1 + 1/4 + 1/9 + 1/16 + \cdots = \pi^2/6$

3. Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the detail.
   (a) $f(x) = \begin{cases} 0, & \text{if } 0 < x < 2 \\ 1, & \text{if } 2 \le x < 4 \end{cases}$;
   (b) $f(x) = \pi - x \quad (0 < x < \pi)$;
   (c) $f(x) = \begin{cases} x, & \text{if } 0 < x < \pi/2 \\ \pi/2, & \text{if } \pi/2 \le x < \pi \end{cases}$;
   (d) $f(x) = x \quad (0 < x < L)$

3 Forced Oscillations

1. Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.
   (a) $r(t) = \sin \alpha t + \sin \beta t,\ \omega^2 \neq \alpha^2, \beta^2$;
   (b) $r(t) = \sin t,\ \omega = 0.5, 0.9, 1.1, 1.5, 10$;
   (c) $r(t) = \pi/4\,|\sin t|$, if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t),\ \ |\omega| \neq 0, 2, 4, \cdots$.
   (d) $r(t) = \begin{cases} -1, & \text{if } -\pi < t < 0 \\ 1, & \text{if } 0 < t < \pi \end{cases}$ and $|\omega| \neq 1, 3, 5, \cdots$;

2. Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details.

(a) $r(t) = \begin{cases} -1, & \text{if } -\pi < t < 0 \\ 1, & \text{if } 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$;

(b) $r(t) = \begin{cases} t, & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t, & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

3. Find the steady-state current $I(t)$ in the RLC-circuit in Fig. 275, where $R = 10\Omega$, $L = 1H$, $C = 10^{-1}F$ and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. Hint. Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

(a) $E(t) = \begin{cases} -50t^2, & \text{if } -\pi < t < 0 \\ 50t^2, & \text{if } 0 < t < \pi \end{cases}$;

(b) $E(t) = \begin{cases} 100(t - t^2), & \text{if } -\pi < t < 0 \\ 100(t + t^2), & \text{if } 0 < t < \pi \end{cases}$.

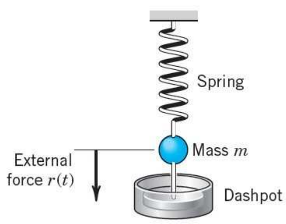

Fig. 274. Vibrating system under consideration

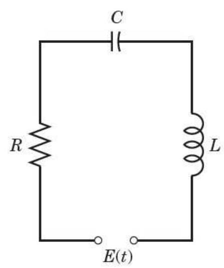

Fig. 275. Electrical analog of the system in Fig. 274 (*RLC-circuit*)