Math 141

Lecture 6

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Outline

1. Functions of Several Variables
   - Verbal description
   - Numerical description
   - Analytical description

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- Variables may be “dependent” or independent - issue dealt with in the subject of probabilities/statistics.
- We need to build and use functions with multidimensional input.
- Such input is typically represented as a bundle of scalar variables.

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- We explain by examples.

Verbal description examples

- The apparent temperature, W, felt on exposed skin depends on several factors, including the actual temperature, T, the wind speed, v, and the humidity. The *wind chill temperature* is a mathematical model for W under the assumption that the humidity is 0 and that the only factors influencing W are T and v:

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  $$W = W(T, v) \, .$$

  The domain of the function W consists of all reasonable pairs (T, v).
- The Cobb-Douglas production function models the production output, P, under the assumption that the only factors are the amount of labor, L, and the amount of capital, K:

  $$P = P(L, K) \, .$$

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- A set (ρ, ϕ, θ) of spherical coordinates determines the rectangular coordinates (x, y, z) of a point. In this case, both the input and the output are multidimensional:

$$(x, y, z) = \mathbf{F}(\rho, \theta, \phi) \ ,$$

- The wind velocity **v** at a point *P* depends on the position **r** of *P*,

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  In this case both the input and the output are vectorial quantities.
- The electric force on a charge q displaced by $\mathbf{r}$ from a charge Q depends on the two charges, the displacement, and the medium in which the charges are placed:

$$\mathbf{E} = \mathbf{E}(q, Q, \mathbf{r}) \; .$$

  Note that in this case the output data is a vector and the input data is a mix of scalar and vectorial quantities.

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- This table contains numerical data collected through experiments at selected input levels.

Examples: describing multivariable function via numerical data.

- the following is Wind Chill Chart provided by NOOA:

NEW NWS WIND CHILL CHART

Temperature (°F)

Wind (mph) / Calm	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5	36	31	25	19	13	7	1	-5	-11	-16	-22	-28	-34	-40	-46	-52	-57	-63
10	34	27	21	15	9	3	-4	-10	-16	-22	-28	-35	-41	-47	-53	-59	-66	-72
15	32	25	19	13	6	0	-7	-13	-19	-26	-32	-39	-45	-51	-58	-64	-71	-77
20	30	24	17	11	4	-2	-9	-15	-22	-29	-35	-42	-48	-55	-61	-68	-74	-81
25	29	23	16	9	3	-4	-11	-17	-24	-31	-37	-44	-51	-58	-64	-71	-78	-84
30	28	22	15	8	1	-5	-12	-19	-26	-33	-39	-46	-53	-60	-67	-73	-80	-87
35	28	21	14	7	0	-7	-14	-21	-27	-34	-41	-48	-55	-62	-69	-76	-82	-89
40	27	20	13	6	-1	-8	-15	-22	-29	-36	-43	-50	-57	-64	-71	-78	-84	-91
45	26	19	12	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79	-86	-93
50	26	19	12	4	-3	-10	-17	-24	-31	-38	-45	-52	-60	-67	-74	-81	-88	-95
55	25	18	11	4	-3	-11	-18	-25	-32	-39	-46	-54	-61	-68	-75	-82	-89	-97
60	25	17	10	3	-4	-11	-19	-26	-33	-40	-48	-55	-62	-69	-76	-84	-91	-98

FROSTBITE OCCURS IN 15 MINUTES OR LESS

- Another example is the Income Tax Table. Explain what the input and output variables are in that case.

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with W and T in Fahrenheit and v in *mph*.

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- The above terms are not precisely defined and not fully agreed upon.

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- The transition formulas from spherical to rectangular coordinates are a derived via geometric reasoning.

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- Polynomials of degree two in two and three variables are parametrized by:

$$\begin{aligned} f(x, y) &= a_{11}x^2 + a_{12}xy + a_{22}y^2 + a_1x + a_2y + a_0 \\ g(x, y, z) &= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz + \\ &\quad + a_1x + a_2y + a_3z + a_0 \end{aligned}$$

where the a_{ij}'s are real numbers.

- The formula for electric force is given by laws of physics: the magnitude of the force is directly proportional to the charges q, Q, and inversely proportional to the square of the distance between them. The force acts along the line joining the two points, attracts q to Q if the charges have different sign and rejects q from Q if the charges have the same sign. The mathematical translation is

$$\mathbf{E}(q, Q, \mathbf{r}, \epsilon) = \frac{\epsilon q Q}{|\mathbf{r}|^3}\mathbf{r} \ ,$$

where ϵ is a proportionality constant, depending on the medium the charges are placed in.

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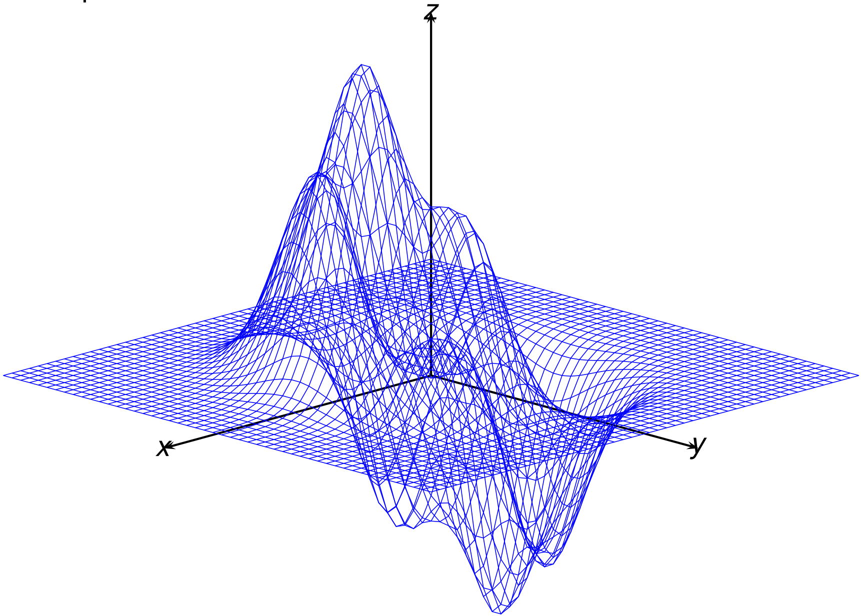
- Even so, “a picture is worth a thousand words (and, say, 10 f-las)”.

Graph of a scalar function.



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- The *graph* of the function $f\colon D \to \mathbb{R}$, where D is a region in $\mathbb{R}^2$, is the set of points $P(x, y, z)$ in $\mathbb{R}^3$ whose coordinates satisfy the condition

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$$z = f(x, y) \quad .$$
- For example, the graph of $f(x, y) = 2x - y + 3$ is the set
$$\{(x, y, z) \,|\, z = 2x - y + 3\} \implies \text{plane } 2x - y - z + 3 = 0 \ .$$

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- Γ = points in $\mathbb{R}^3$ such that $z = x^2 + 2y^2$. The set is not a plane: what does it look like?
- To answer look at sections. Use imaginary CT scan to cut the graph; assemble resulting sections into a graph.

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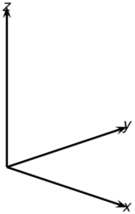

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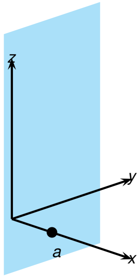

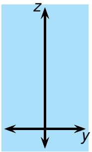

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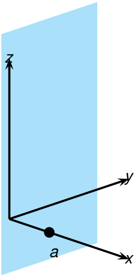

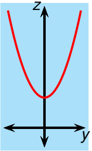

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- In other words, treat x as constant and study the f-n $y \to z = a^2 + 2y^2 = g(a, y)$.

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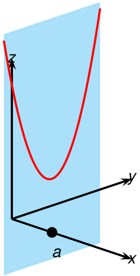

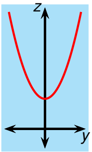

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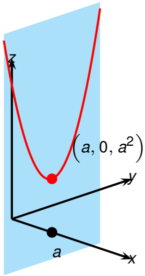

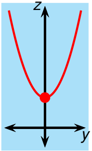

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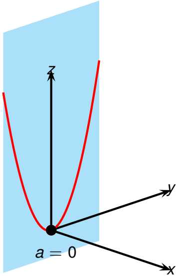

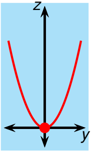

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- As a moves away from 0, the parabola vertex rises.

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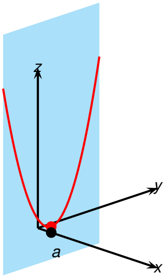

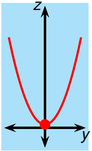

The plane $x = a$.

- Cut by vertical planes $x = a$, a-constant, parallel to the $Oyz-$plane.
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- The cross-sections are the curves: $\{(a,y,z) \quad \text{where } z = a^2 + 2y^2\}$
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- As a moves away from 0, the parabola vertex rises.

A motivating example

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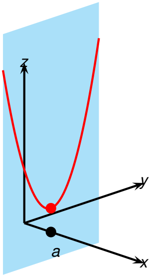

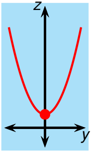

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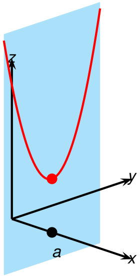

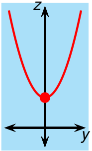

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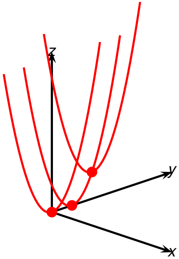

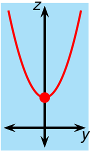

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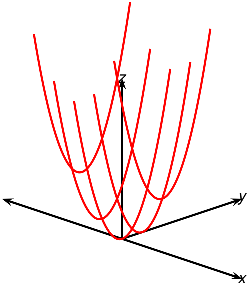

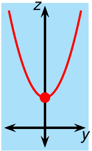

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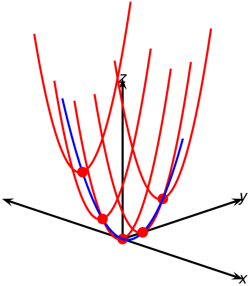

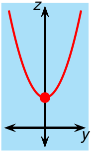

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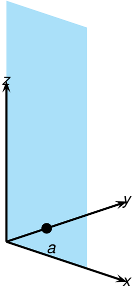

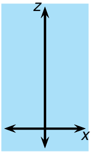

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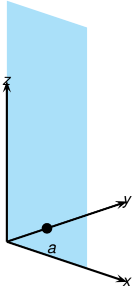

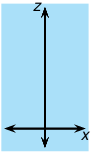

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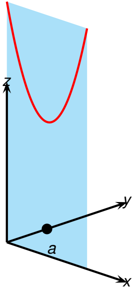

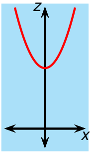

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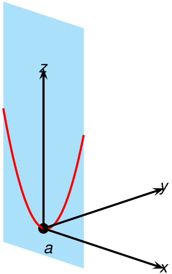

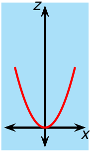

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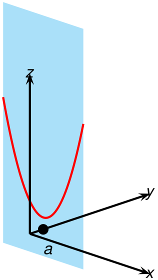

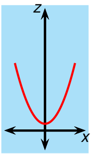

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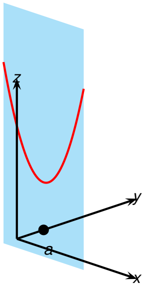

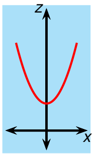

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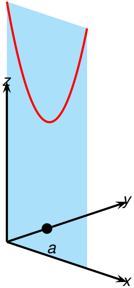

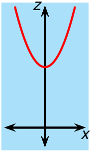

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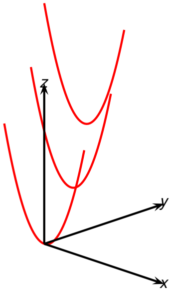

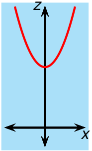

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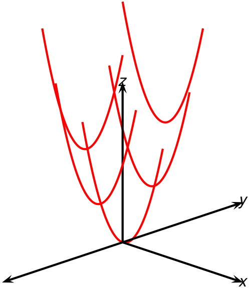

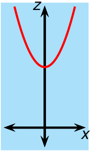

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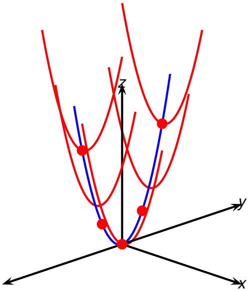

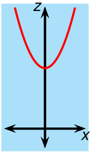

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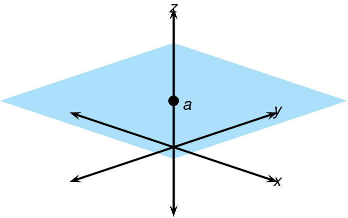
- The vertices are rising as we move away from the origin.

A motivating example

$g(x, y) = x^2 + 2y^2$



- For horizontal sections keep constant the output variable, $z = a$.

A motivating example

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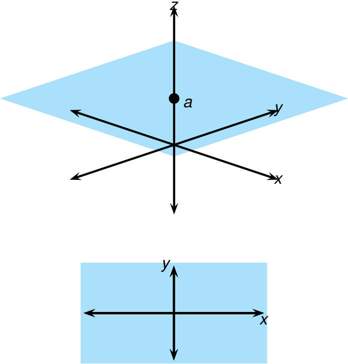

$x^2 + 2y^2 = a$, $a < 0$.

- For horizontal sections keep constant the output variable, $z = a$.
- When we intersect with $z = a$ we get the curve with equations $x^2 + 2y^2 = a$, $z = a$.

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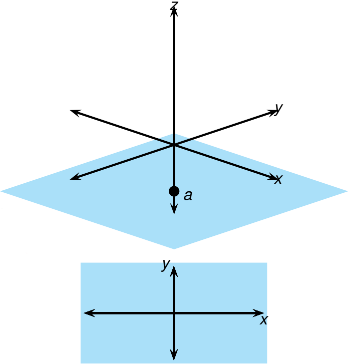

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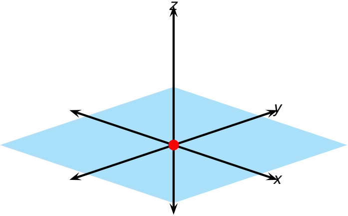

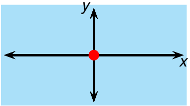

$x^2 + 2y^2 = 0, .$

- For horizontal sections keep constant the output variable, $z = a$.
- When we intersect with $z = a$ we get the curve with equations $x^2 + 2y^2 = a$, $z = a$.
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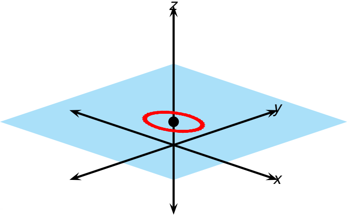

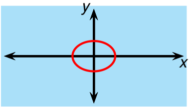

$x^2 + 2y^2 =$, $a > 0$.

- For horizontal sections keep constant the output variable, $z = a$.
- When we intersect with $z = a$ we get the curve with equations $x^2 + 2y^2 = a$, $z = a$.
- For $a < 0$ intersection is empty.
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- For $a > 0$ intersection is an ellipse.

A motivating example

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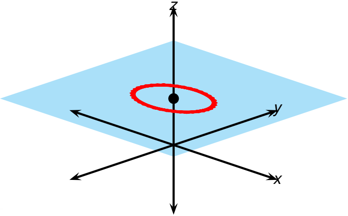

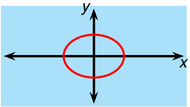

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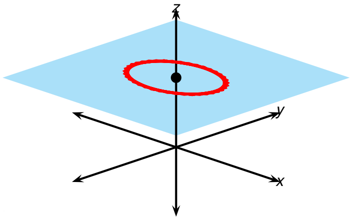

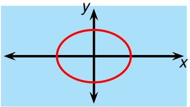

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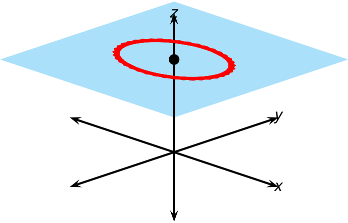

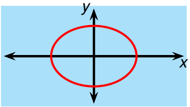

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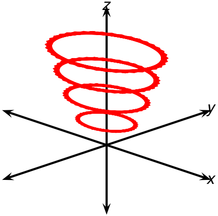

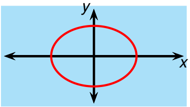

$x^2 + 2y^2 = , a > 0.$

- For horizontal sections keep constant the output variable, $z = a$.
- When we intersect with $z = a$ we get the curve with equations $x^2 + 2y^2 = a$, $z = a$.
- For $a < 0$ intersection is empty.
- For $a = 0$ intersection is $(0, 0, 0)$.
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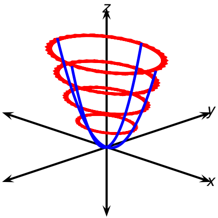

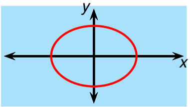

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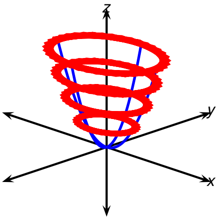

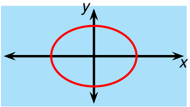

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Definition

The sets $\{(x, y, a) | g(x, y) = a\}$ are called level curves of the function g.

- You should be familiarized with level curves if you have ever seen a topographic map or from weather reports on the tv.
- What are the functions in those cases?

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- In this way we represent the f-n graphically using dimension equal to the number of input variables.

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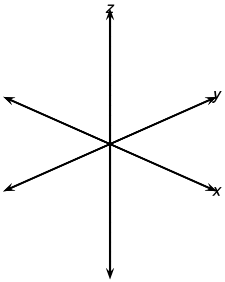

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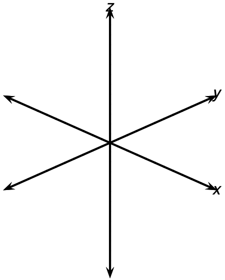

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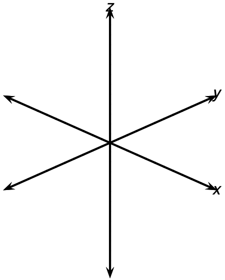

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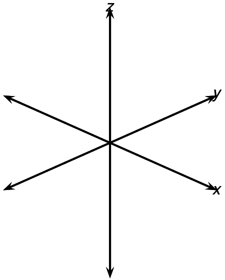

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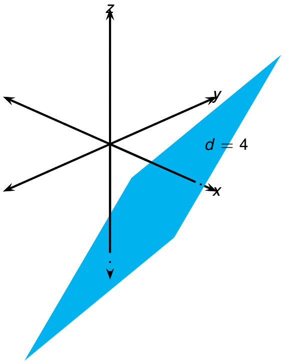

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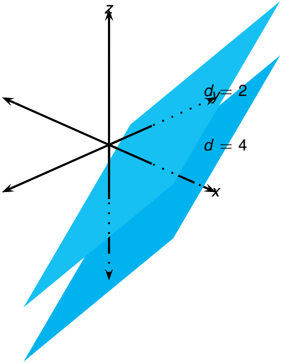

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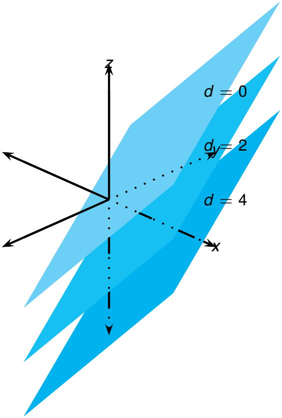

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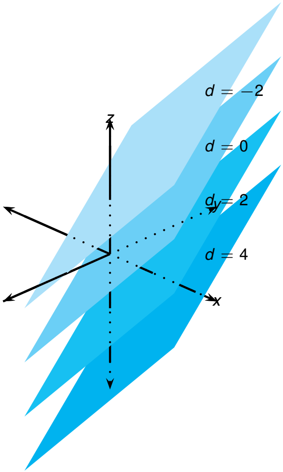

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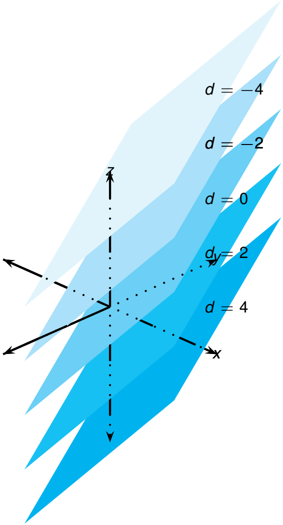

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To understand surfaces in space we need the following.

Remark

The level set $f(x, y, z) = 0$ for the function

$$f(x, y, z) = ax + by - z + d$$

is the same as the graph of the function $g(x, y) = ax + by + c$.

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- We'll show that under reasonable assumptions, level surfaces can *locally* be described as graph surfaces.

Vector fields

- *Vector fields* are functions with multidimensional input and output.
- Input is point in space; output is a vector, which we plot as a vector with a tail at the input point.
- Examples
  - Velocity of fluid/air at given point;
  - Electric force per unit of charge;
  - Gravitational field;

Coordinate representation of vector fields

- In rectangular coordinates a vector field $\mathbf{F}$ can be decomposed along the fundamental directions:

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k} \; .$$

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- For regions in the plane 2-dim vector fields are defined in a similar fashion: as function from subsets of $\mathbb{R}^2$ to $\mathbb{R}$:

$$\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$$

- Example: define the vector field $\mathbf{e}_r$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ via $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} = \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$

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- Similarly define the vector field $\mathbf{e}_\theta$ by:
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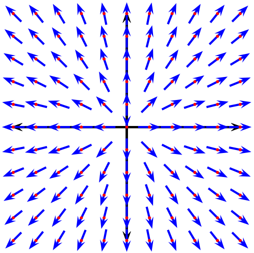

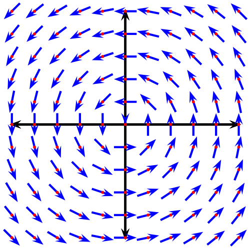

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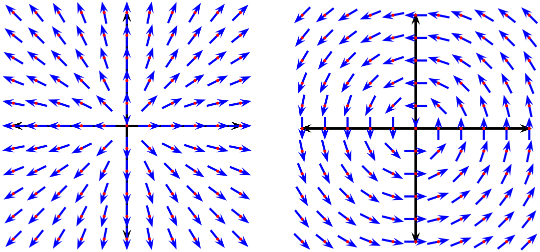

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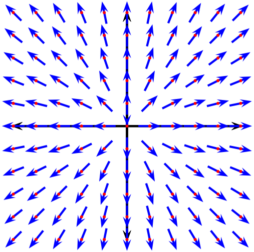

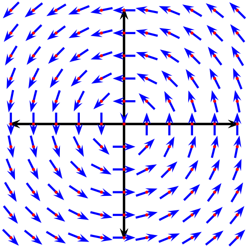

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- By "flowing" we mean an object whose velocity at each point is given by the value of the field.