Math 141

Lecture 7

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2014

Outline

1. Surfaces
   - Implicit functions
   - Parametrizations
   - Quadric Surfaces

- A surface S can be given in two different ways:

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- A graph surface $z = f(x, y)$ can always be represented as a level surface:

$$z = f(x, y) \Longleftrightarrow F(x, y, z) = 0 \quad \text{for} \quad F(x, y, z) = z - f(x, y) \, .$$

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- Locally, with additional requirements, a level surface can be represented as graph surfaces.
    - Near $P(0, 0, 1)$, the surface is the graph surface of $z = \sqrt{1 - x^2 - y^2}$.
    - Near $P(1, 0, 0)$, the surface is not a graph surface w.r.t. to z.

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  - $F(x, y, f(x, y)) = 0$ for all (x, y) in the disk D.
- If the level surface is a graph surface, we say that the equation $F(x, y, z) = k$ implicitly defines $z = f(x, y)$ satisfying the condition $f(x_0, y_0) = z_0$.

- The equation $x^2 + y^2 + z^2 = 1$ implicitly defines $z = \sqrt{1 - x^2 - y^2}$ as the unique function $z = f(x, y)$ such that
  - $x^2 + y^2 + (f(x, y))^2 = 1$ for all (x, y) in a disk around $(0, 0)$;
  - $f(0, 0) = 1$.
- The equation $x^2 + y^2 + z^2 = 1$ implicitly defines $z = -\sqrt{1 - x^2 - y^2}$ as the unique function $z = f(x, y)$ such that
  - $x^2 + y^2 + (f(x, y))^2 = 1$ for all (x, y) in a disk around $(0, 0)$;
  - $f(0, 0) = -1$.

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- The function $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{f}(s,t) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$ globally identifies $\mathbb{R}^2$ with the plane $\mathcal{P}$.

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- Each point of $\mathcal{P}$ corresponds uniquely to pair (s,t).

Example: parametrization of part of sphere

- Consider $f\colon (0,2\pi)\times(0,\pi)\to\mathbb{R}^3$ given by

$$f(t,s) = (R\sin s\cos t, R\sin s\sin t, R\cos s)\ .$$

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- The image of f is a subset of the sphere of radius R centered at the origin.
- Which points of the sphere are not covered by the image of f?

Quadratic surfaces

- The level sets for second degree polynomial functions

  $$f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J.$$

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- The coefficients are allowed to be zero.

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- $$\begin{aligned} Ax^2 + By^2 + Cz^2 + D &= 0, \\ A(-x)^2 + B(-y)^2 + C(-z)^2 + D &= 0, \end{aligned}$$

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The canonical forms above are in addition split into sub-forms depending on the sign of A, B, C, D, I.

$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0, B > 0, C > 0, D > 0$

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- Example:

$$x^2 + 2y^2 + 3z^2 + 4 = 0.$$

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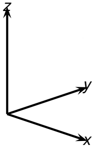
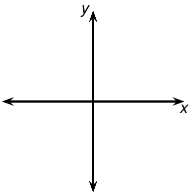


- Let $A > 0, B > 0, C > 0$, $D < 0$. Rescale so $D = -1$.



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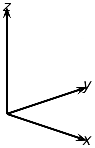

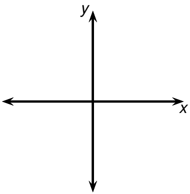

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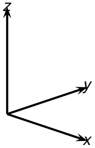

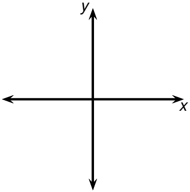

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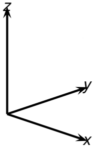

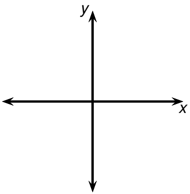

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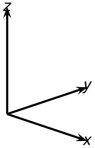

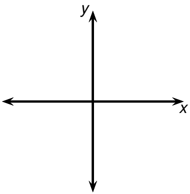

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- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:

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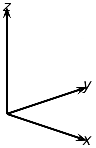

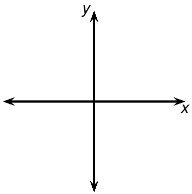

$z =$ -3

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

$=$

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- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
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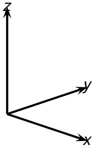

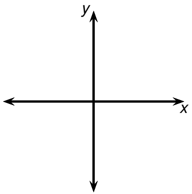

$z =$ -3

$$\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4} = -\frac{5}{4}$$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
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- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.

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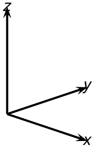

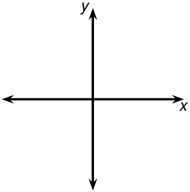

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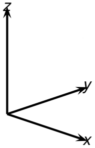
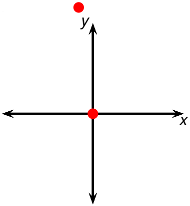
$=$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





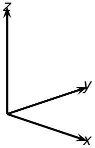
$z = -2$

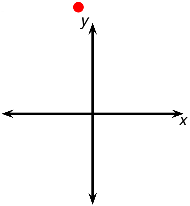
$\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$
$= 0$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}, B = \frac{1}{b^2}, C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.

$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = -1$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

$=$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
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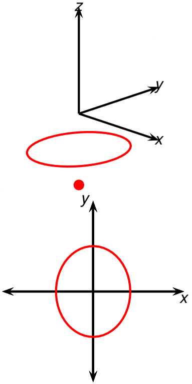
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.



$z = -1$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4} = \frac{3}{4}$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
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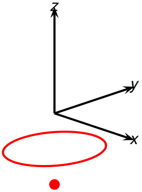
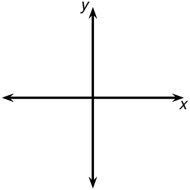
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = 0$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

$=$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
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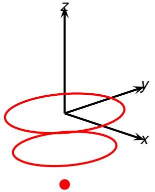
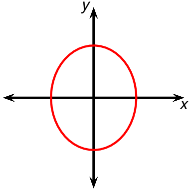
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





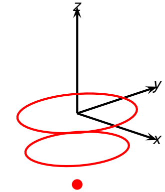
$z = 0$

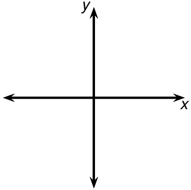
$$\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4} = 1$$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}, B = \frac{1}{b^2}, C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = 1$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

$=$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) \middle| \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
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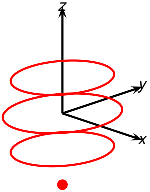
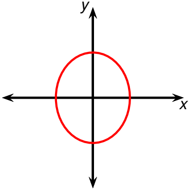
$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = 1$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$

$= \frac{3}{4}$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
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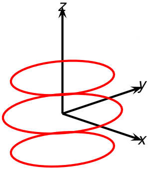
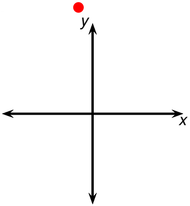
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = 2$

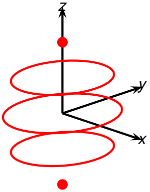
$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

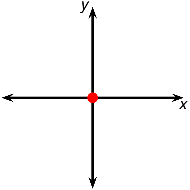
$=$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z = 2$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

$= 0$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
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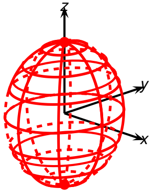
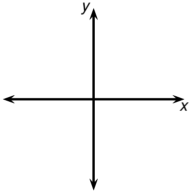
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C > 0, D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.





$z =$

$\frac{1}{2}x^2 + \frac{1}{3}y^2 \quad = 1 - \frac{z^2}{4}$

- Let $A > 0, B > 0, C > 0, D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}, B = \frac{1}{b^2}, C = \frac{1}{c^2}$.
- Surface becomes: $\left\{(x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.
- Figure is called ellipsoid.

Summary: surfaces of form $Ax^2 + By^2 + Cz^2 + D = 0$

A	B	C	D	$x = x_0$	$y = y_0$	$z = z_0$	Example	Name
> 0	> 0	> 0	> 0	empty	empty	empty	$x^2 + 2y^2 + 3z^2 + 4 = 0$	empty
> 0	> 0	> 0	$= 0$					
> 0	> 0	> 0	< 0	ellipse	ellipse	ellipse	$x^2 + 2y^2 + 3z^2 - 4 = 0$	Ellipsoid
> 0	> 0	$= 0$	> 0					
> 0	> 0	$= 0$	$= 0$					
> 0	> 0	$= 0$	< 0					
> 0	> 0	< 0	> 0					
> 0	> 0	< 0	$= 0$					
> 0	> 0	< 0	< 0					
> 0	$= 0$	$= 0$	> 0					
> 0	$= 0$	$= 0$	$= 0$					
> 0	$= 0$	$= 0$	< 0					

Fill in the rest of the table.

Quadratics $Ax^2 + By^2 + Iz = 0$ (no central symmetry)

$$Ax^2 + By^2 + Iz = 0$$

A	B	$x = x_0$	$y = y_0$	$z = z_0$	Example	Name
> 0	> 0	parabola	parabola	ellipse, point, or empty	$x^2 + 2y^2 + 3z = 0$	Elliptic paraboloid
> 0	$= 0$					
> 0	< 0					

Fill in the rest of the table.