Matrix Representation of Graph

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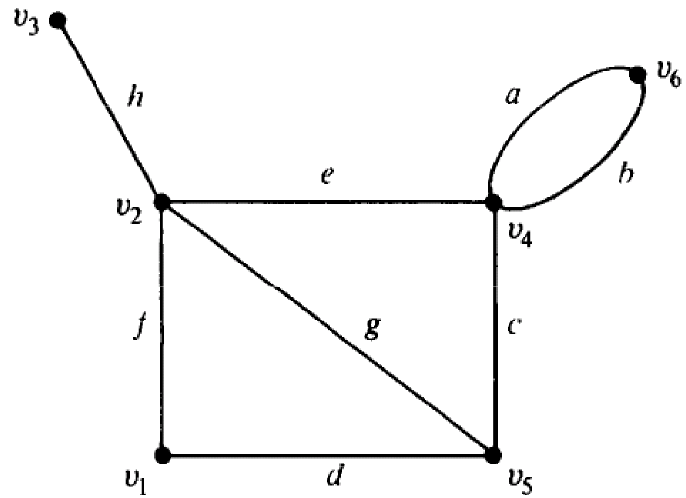
Incidence Matrix

The matrix element

$$a_{ij} = 1, \quad \text{if } j\text{th edge } e_j \text{ is incident}$$
$$= 0, \quad \text{otherwise.}$$



	a	b	c	d	e	f	g	h
v_1	0	0	0	1	0	1	0	0
v_2	0	0	0	0	1	1	1	1
v_3	0	0	0	0	0	0	0	1
v_4	1	1	1	0	1	0	0	0
v_5	0	0	1	1	0	0	1	0
v_6	1	1	0	0	0	0	0	0

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example, columns 1 and 2 in Fig. 7-1.
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $A(G)$ of graph G can be written in a block-diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right], \tag{7-1}$$

THEOREM 7-1

Two graphs G_1 and G_2 are isomorphic if and only
$A(G_1)$ and $A(G_2)$ differ only by permutations of rows

THEOREM 7-2

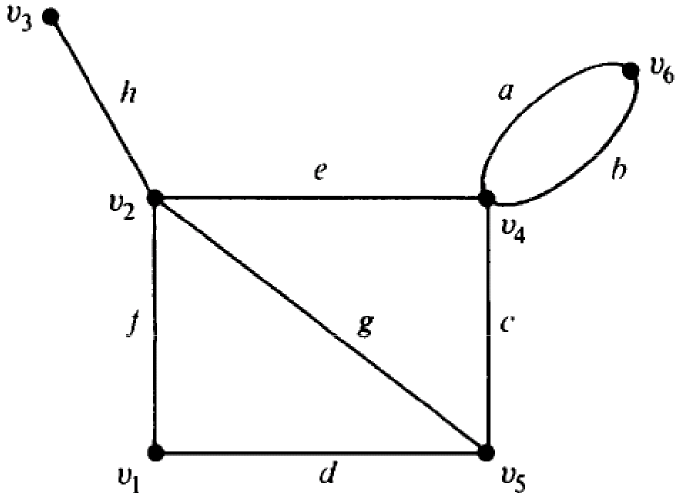
If $A(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $A(G)$ is $n - 1$.

Circuit Matrix

7-3. CIRCUIT MATRIX

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $\mathsf{B} = [b_{ij}]$ of G is a q by e, (0, 1)-matrix defined as follows:

$$
\begin{aligned}
b_{ij} &= 1, \qquad \text{if } i\text{th circuit includes } j\text{th edge, and} \\
&= 0, \qquad \text{otherwise.}
\end{aligned}
$$



$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of B(G) is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self-loop—the corresponding row will have a single 1.
4. The number of 1's in a row is equal to the corresponding circuit.
5. If graph G is separable (or disconnect (or components) g_1 and g_2, the circuit a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & \\ \hline 0 & E \end{array}\right]$$

6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.
7. Two graphs G_1 and G_2 will have the same circuit matrix if and only if G_1 and G_2 are 2-isomorphic (Theorem 4-15). In other words, (unlike

THEOREM 7-4

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

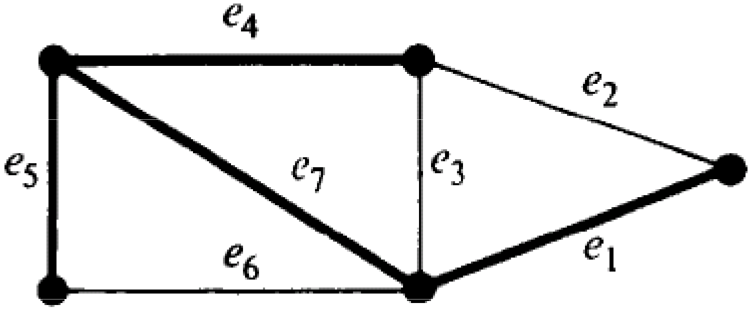
$$A \cdot B^T = B \cdot A^T = 0 \pmod 2, \tag{7-4}$$

Proof: Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.

Fundamental Circuit Matrix

A submatrix (of a circuit matrix) in which all rows corres
fundamental circuits is called a *fundamental circuit matrix* B

As in matrices A and B, permutations of
affect B_f. If n is the number of vertices and e t
ed graph, then B_f is an $(e - n + 1)$ by e
fundamental circuits is $e - n + 1$, each fun
by one chord.



$$
\begin{array}{ccc|cccc}
e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7 \\
\end{array}
$$

$$
\left[\begin{array}{ccc|cccc}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
$$

A matrix B_f thus arranged can be written as

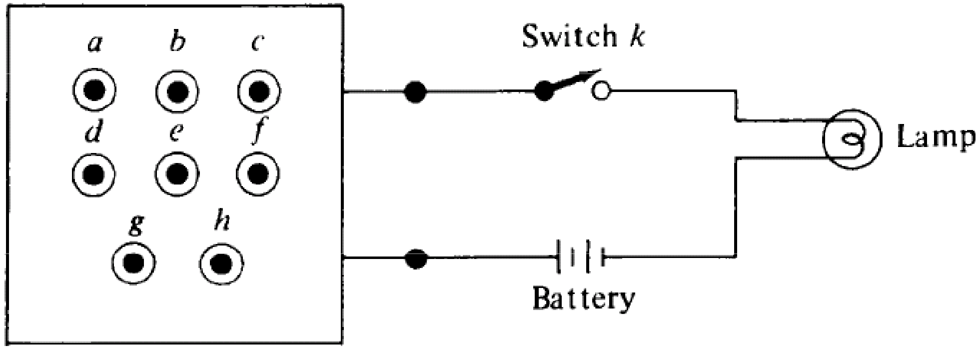
$$B_f = [I_\mu \,\vdots\, B_t], \tag{7-5}$$

where I_μ is an identity matrix of order $\mu = e - n + 1$, and B_t is the remaining μ by $(n - 1)$ submatrix, corresponding to the branches of the spanning tree.

From Eq. (7-5) it is clear that the

$$\text{rank of } B_f = \mu = e - n + 1.$$

Application to Switching Network



The box

(a, b, f, h, k), (a, b, g, k), (a, e, f, g, k), (a, e, h, k),

(b, c, e, h, k), (c, f, h, k), (c, g, k), (d, k).

Solution: Consider the switching network as a graph whose edges represent switches. We can assume that the graph is connected, and has no self-loop. Since a lit lamp implies the formation of a circuit, we can regard the preceding list as a partial list of circuits in the corresponding graph. With this list we form a circuit matrix:

$$
B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & k \end{array} \\
\left[\begin{array}{ccccccccc}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \text{[illegible]} & \text{[illegible]}
\end{array}\right]
\end{array}
$$

Therefore, we can delete the first, third, and fifth rows from matrix B, without any loss of information. Remaining is a 5 by 9 matrix B_1:

$$
B_1 = \begin{array}{c}
\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & k \end{array} \\
\left[\begin{array}{ccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}.
$$

Adding the fourth row in B_2 to the first, we get B_3.

number of edges E=9

nullity Mu=4

rank R=4

number of vertices N=5

$$B_3 = \begin{array}{c} \begin{matrix} b & e & f & g & d & a & c & h & k \end{matrix} \\ \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} = [I_5 \mid F]$$

Since,

$$B_3 = [I_5 \mid F],$$

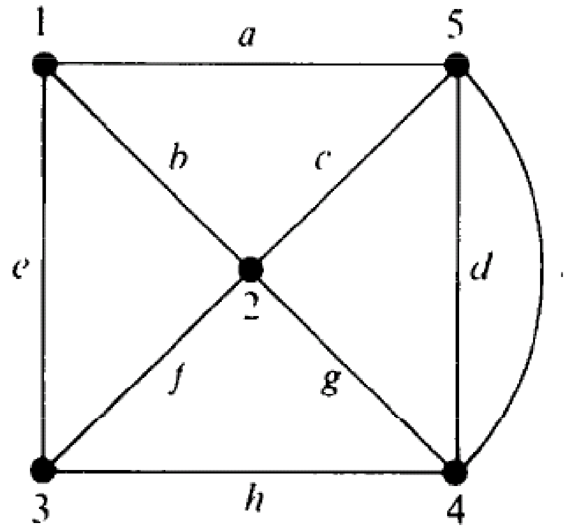
an orthogonal matrix to B_3 is

$$M = [-F^T \mid I_4] = [F^T \mid I_4],$$

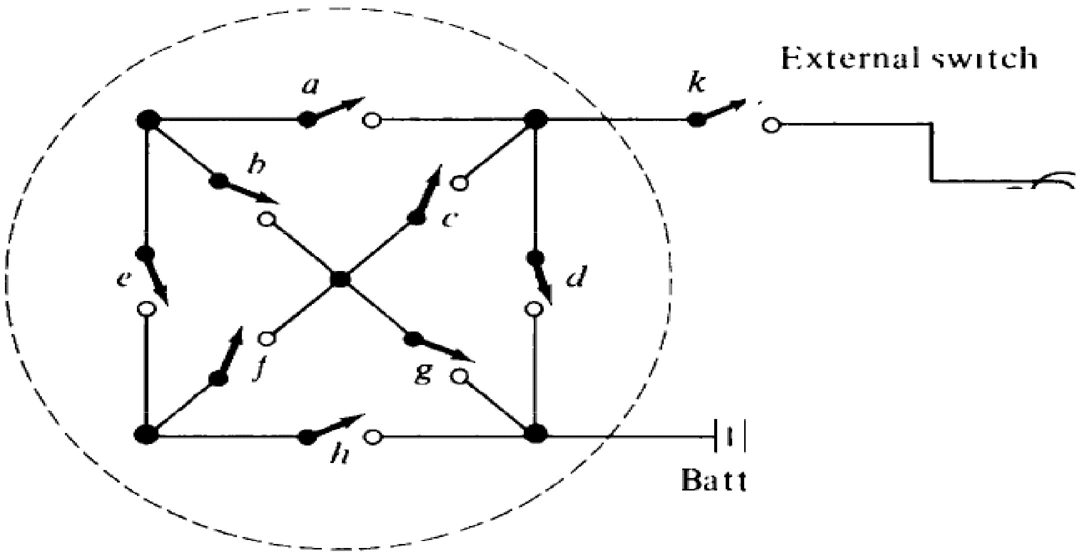
$$M = \begin{array}{c} \begin{matrix} b & e & f & g & d & a & c & h & k \end{matrix} \\ \left[\begin{array}{ccccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$
\mathbf{A} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{ccccccccc}
b & e & f & g & d & a & c & h & k \\
\end{array}
$$

$$
\mathbf{A} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
$$



Inside Black Box

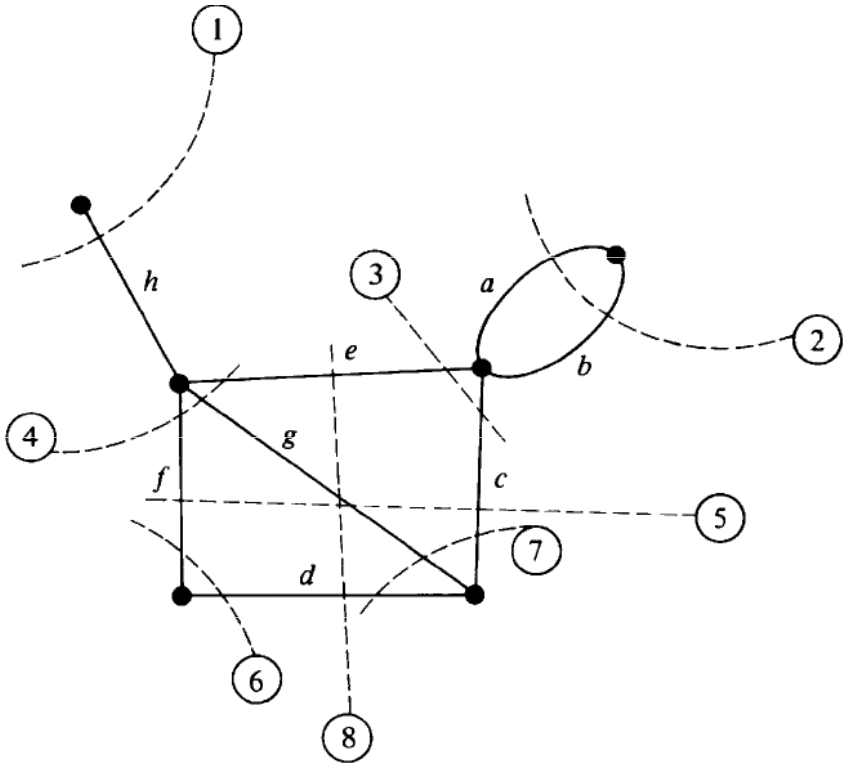


Cut-Set Matrix

$$c_{ij} = 1, \quad \text{if } i\text{th cut-set contains } j\text{th edge, and}$$
$$= 0, \quad \text{otherwise.}$$

1. As in the case of the incidence matrix, a permutation of rows or columns in a cut-set matrix corresponds simply to a renaming of the cut-sets and edges, respectively.
2. Each row in $\mathsf{C}(G)$ is a cut-set vector.
3. A column with all 0's corresponds to an edge forming a self-loop.
4. Parallel edges produce identical columns in the cut-set matrix (e.g., first two columns in Fig. 7-5).
5. In a nonseparable graph, every set of edges incident on a vertex is a cut-set (Problem 4-8). Therefore, every row of incidence matrix $\mathsf{A}(G)$ is included as a row in the cut-set matrix $\mathsf{C}(G)$. In other words, for a

$$\mathsf{B} \cdot \mathsf{C}^T = \mathsf{C} \cdot \mathsf{B}^T = 0 \qquad \text{(mc}$$



C =

	a	b	c	d	e	f	g	h
1	0	0	0	0	0	0	0	1
2	1	1	0	0	0	0	0	0
3	0	0	1	0	1	0	0	0
4	0	0	0	0	1	1	1	0
5	0	0	1	0	0	1	1	0
6	0	0	0	1	0	1	0	0
7	0	0	1	1	0	0	[illegible]	0
8	0	0	0	1	1	[illegible]	1	0

THEOREM 7-6

The rank of cut-set matrix C(G) is equal to the rank of the incidence matrix A(G), which equals the rank of graph G.

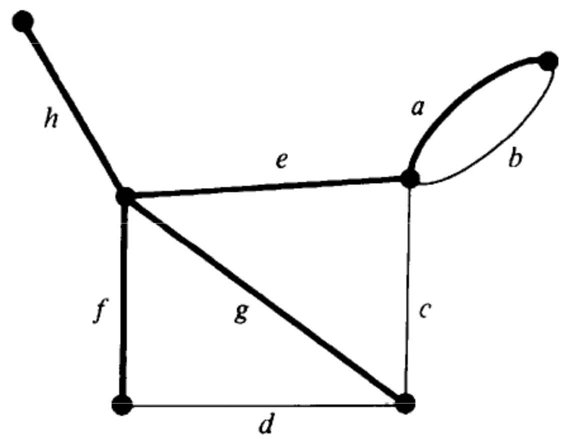
As in the case of a fundamental circuit matrix, a fundamental cut-set matrix $\mathbf{C}_f$ can also be partitioned into two submatrices, one of which is an identity matrix $\mathbf{I}_{n-1}$ of order $n - 1$. That is,

$$\mathbf{C}_f = [\mathbf{C}_c \mid \mathbf{I}_{n-1}], \tag{7-9}$$

Relationship between A_f, B_f and C_f

$$B_f = [I_\mu \mid B_t],$$

$$C_f = [C_c \mid I_{n-1}]$$



$C_f =$

b	c	d	a	e	f	g	h
1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	0
0	0	1	0	0	1	0	0
0	1	1	0	0	0	1	0
0	0	0	0	0	0	0	1

$$A_f = [A_c \mid A_t],$$

$$\mathbf{A}_f \cdot \mathbf{B}_f^T = 0$$

$$[\mathbf{A}_c \mid \mathbf{A}_t] \cdot \begin{bmatrix} \mathbf{I}_\mu \\ \hline \mathbf{B}_t^T \end{bmatrix} = 0,$$

$$\mathbf{A}_c + \mathbf{A}_t \cdot \mathbf{B}_t^T = 0.$$

$$\mathbf{A}_t^{-1} \cdot \mathbf{A}_c = -\mathbf{B}_t^T.$$

Since in mod 2 arithmetic $-1 = 1$,

$$\mathbf{B}_t^T = \mathbf{A}_t^{-1} \cdot \mathbf{A}_c.$$

Similarly, since the columns in $\mathbf{B}_f$ and $\mathbf{C}_f$ are arranged in the same order, according to Eq. (7-4), we have (in mod 2 arithmetic)

$$\mathbf{C}_f \cdot \mathbf{B}_f^T = 0.$$

$$[\mathbf{C}_c \mid \mathbf{I}_{n-1}] \cdot \begin{bmatrix} \mathbf{I}_\mu \\ \hline \mathbf{B}_t^T \end{bmatrix} = 0,$$

$$\begin{aligned} \mathbf{C}_c &= -\mathbf{B}_t^T \\ &= \mathbf{B}_t^T \\ &= \mathbf{A}_t^{-1} \cdot \mathbf{A}_c, \end{aligned}$$

$$\mathsf{A}_f = \begin{array}{c} \begin{array}{cccccccc} b & c & d & a & e & f & g & h \end{array} \\ \left[\begin{array}{ccc:ccccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} = [\mathsf{A}_c \,\vdots\, \mathsf{A}_t],$$

$$\mathsf{B}_f = \begin{array}{c} \begin{array}{cccccccc} b & c & d & a & e & f & g & h \end{array} \\ \left[\begin{array}{ccc:ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & \text{[illegible]} \end{array}\right] \end{array}$$

$$\mathsf{C}_f = \begin{array}{c} \begin{array}{cccccccc} b & c & d & a & e & f & g & \text{[illegible]} \end{array} \\ \left[\begin{array}{ccc:ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & \text{[illegible]} \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \text{[illegible]} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & \text{[illegible]} \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

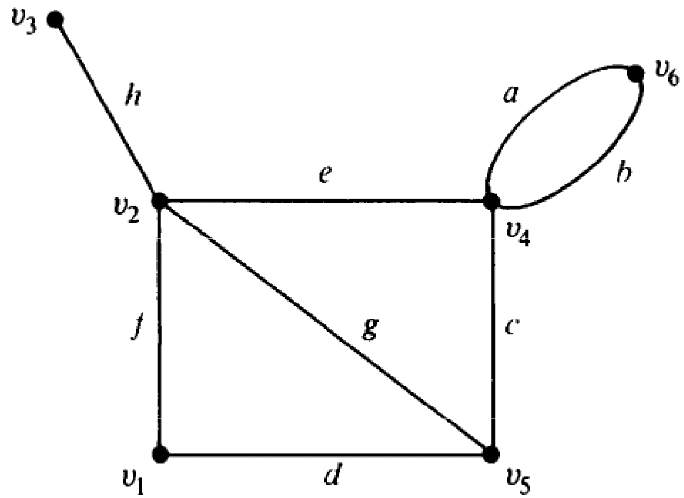
$\mathsf{B}_t^T = \mathsf{C}_c$ is immediate. It can also be readily verified that

$$\mathsf{A}_t^{-1} \cdot \mathsf{A}_c = \mathsf{B}_t^T.$$

Path Matrix

$$p_{ij} = 1, \quad \text{if } j\text{th edge lies in } i\text{th path, and}$$
$$\quad\;\; = 0, \quad \text{otherwise.}$$

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



1. A column of all 0's corresponds to ar path between x and y.
2. A column of all 1's corresponds to a between x and y.
3. There is no row with all 0's.

THEOREM 7-7

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix P(x, y), then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

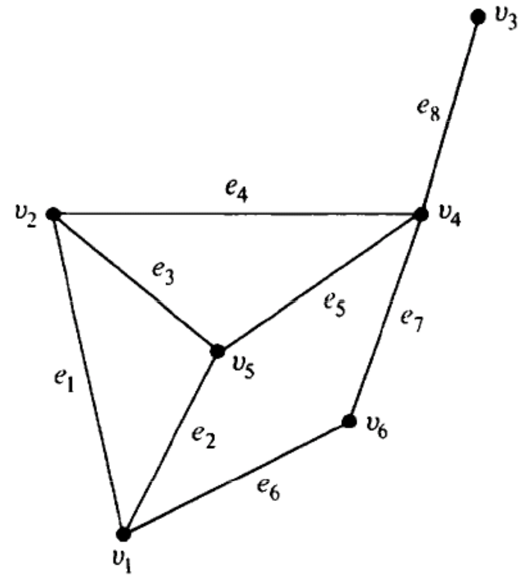
where the matrix M has 1's in two rows x and y, and the rest of the $n - 2$ rows are all 0's.

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{(mod 2).}$$

Adjacency Matrix

$$x_{ij} = 1, \quad \text{if there is an edge between } i\text{th}$$
$$= 0, \quad \text{if there is no edge between the}$$



X =

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	1	1
v_2	1	0	0	1	1	0
v_3	0	0	0	1	0	0
v_4	0	1	1	0	1	1
v_5	1	1	0	1	0	0
v_6	1	0	0	1	0	0

1. The entries along the principal diago
the graph has no self-loops. A self-loo
$x_{ii} = 1$.
2. The definition of adjacency matrix
edges. This is why the adjacency m
without parallel edges.†
5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $\mathsf{X}(G)$ can be partitioned as

$$\mathsf{X}(G) = \left[\begin{array}{c|c} \mathsf{X}(g_1) & 0 \\ \hline 0 & \mathsf{X}(g_2) \end{array}\right],$$

Powers of X, Path MAtrix

The value of an off-diagonal entry in X^2, th
= number of 1's in the dot product of ith
row) of X.
= number of vertices that are adjacent to bot
= number of different paths of length two betv

$$X^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & \text{[illegible]} \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Ij th entry in X^2

= dot product of ith row X^2 and jth column (or row) of X.

$= \sum_{k=1}^{n}$ ikth entry of $X^2 \cdot kj$th entry of X.

$= \sum_{k=1}^{n}$ number of all different edge sequences† of three edges from ith to jth vertex via kth vertex.

= number of different edge sequences of three edges between ith and jth vertices.

$$X^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}.$$

THEOREM 7-8

Let X be the adjacency matrix of a simple graph G. Then the ijth entry in X^r is the number of different edge sequences of r edges between vertices v_i and v_j.

COROLLARY A

In a connected graph, the distance between two vertices v_i and v_j (for $i \neq j$) is k, if and only if k is the smallest integer for which the i, jth entry in x^k is nonzero.

COROLLARY B

If X is the adjacency matrix of a graph G with n

$$Y = X + X^2 + X^3 + \cdots + X^{n-1}, \qquad \text{(in th}$$

then G is disconnected if and only if there exists at that is zero.

COLORING, COVERING & PARTITIONING

Coloring

Chromatic number

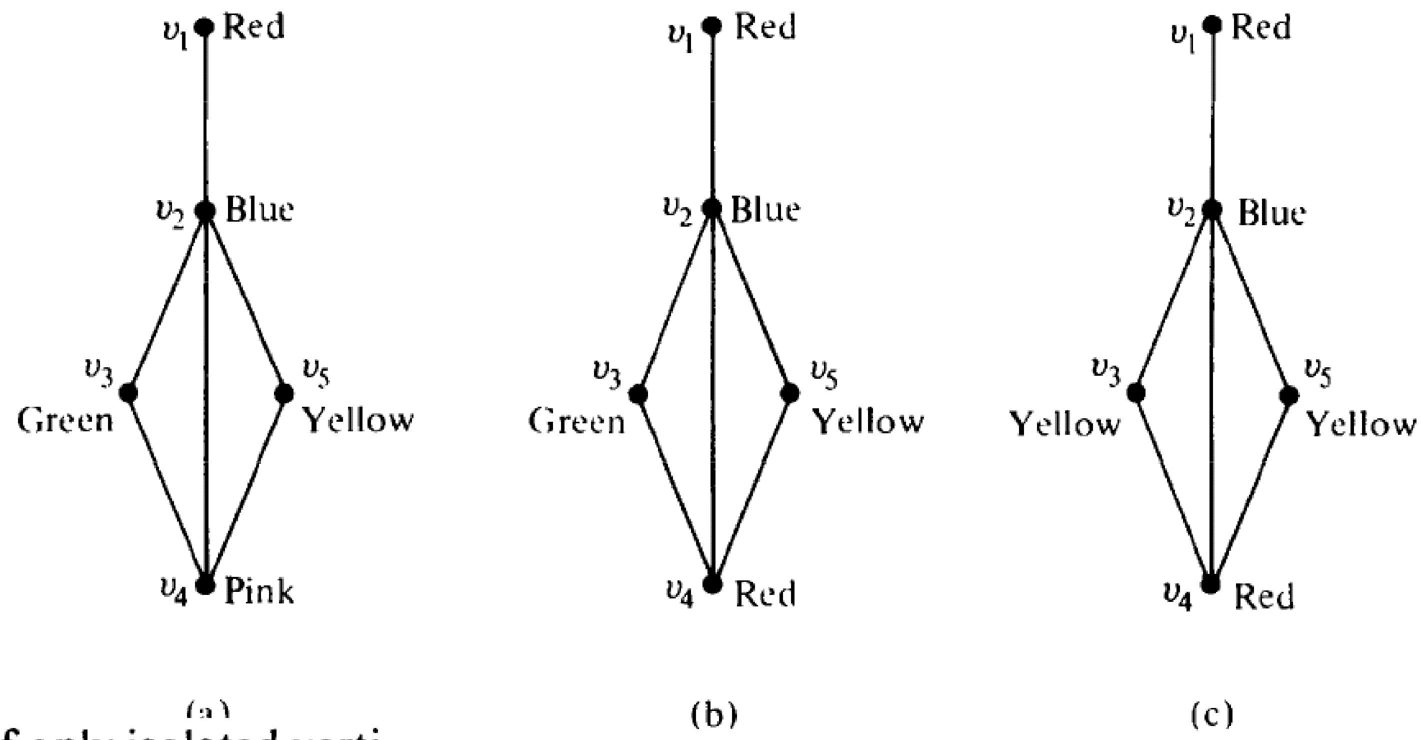


Fig. 8-1 Proper colorings of a graph.

1. A graph consisting of only isolated verti
2. A graph with one or more edges (not a
   2-chromatic (also called *bichromatic*).
3. A complete graph of *n* vertices is *n*-ch
   adjacent. Hence a graph containing a
   at least *r*-chromatic. For instance, ever
   least 3-chromatic.

THEOREM 8-1

Every tree with two or more vertices is 2-chromatic.

THEOREM 8-2

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

THEOREM 8-3

If d_{max} is the maximum degree of the vertices in a graph G,

$$\text{chromatic number of } G \leq 1 + d_{max}.$$

Brooks [8-1] showed that this upper bound can be improved by 1 if G has no complete graph of $d_{max} + 1$ vertices. In that case

$$\text{chromatic number of } G \leq d_{max}.$$

Chromatic Partitioning

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Fig. 8-1(c) produces the partitioning

$$\{v_1, v_4\}, \quad \{v_2\}, \quad \text{and} \quad \{v_3, v_5\}.$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:

A set of vertices in a graph is said to be an *independent set* of vertices or simply an *independent set* (or an *internally stable set*) if no two vertices in the set are adjacent. For example, in Fig. 8-3, $\{a, c, d\}$ is an independent set. A single vertex in any graph constitutes an independent set.

A *maximal independent set* (or *maximal internally stable set*) is an independent set to which no other vertex can be added without destroying its independence property. The set $\{a, c, d, f\}$ in Fig. 8-3 is a maximal indepen-

The number of vertices in the largest independent set of a graph G is called the *independence number* (or *coefficient of internal stability*), $\beta(G)$.

Consider a κ-chromatic graph G of n vertices properly colored with κ different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number $\beta(G)$, we have the inequality

$$\beta(G) \geq \frac{n}{\kappa}.$$

Finding Maximally independent Set

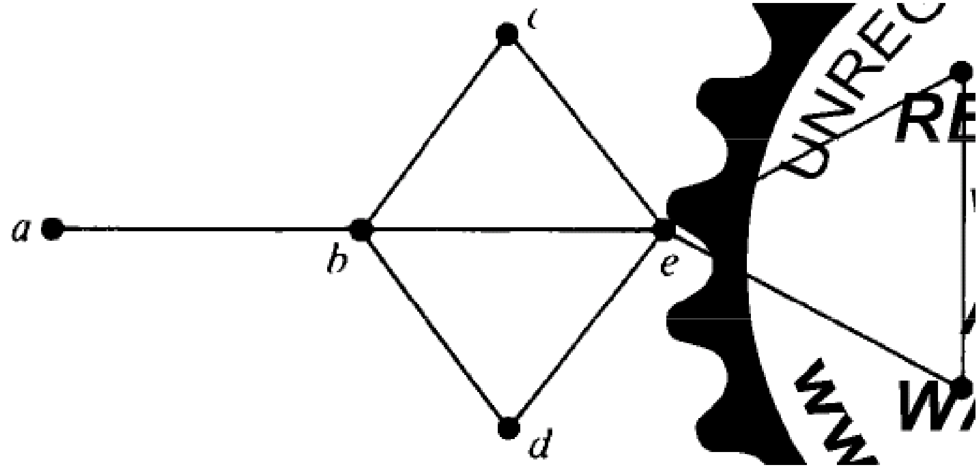
For a given graph G we must find a maximal subset of vertices that does not include the two end vertices of any edge in G. Let us express an edge (x, y) as a Boolean product, xy, of its end vertices x and y, and let us sum all such products in G to get a Boolean expression

$$\varphi = \Sigma\, xy \qquad \text{for all } (x, y) \text{ in } G.$$

$$\varphi' = f_1 + f_2 + \cdots + f_k$$

$$\varphi = ab + bc + bd + be + ce + de + ef + eg + fg,$$

$$\varphi' = (a' + b')(b' + c')(b' + d')(b' + e')(c' + e')(d' + e')$$
$$(e' + f')(e' + g')(f' + g').$$



$$aa = a,$$

$$a + a = a,$$

$$a + ab = a,$$

$$\varphi' = b'e'f' + b'e'g' + a'c'd'e'f' + a'c'd'e'g' + b'c'd'f'g'.$$

acdf, *acdg*, *bg*, *bf*, and *ae*.

Chromatic Partitioning

Dominating Sets: A *dominating set* (or an *externally stable set*) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set. For instance, the vertex set $\{b, g\}$ is a

5. Every maximal independent set is a dominating set.

6. An independent set has the dominance property only if it is a maximal independent set. Thus an *independent dominating set* is the same as a maximal independent set.

7. In any graph G,

$$\alpha(G) \leq \beta(G).$$

Finding Minimal Dominating Set

Finding Minimal Dominating Sets: A method for obtaining all minimal dominating sets in a graph will now be developed. The method, like the one for finding all maximal independent sets, also uses Boolean arithmetic.

To dominate a vertex v_i we must either include v_i or any of the vertices adjacent to v_i. A minimum set satisfying this condition for every vertex v_i is a desired set. Therefore, for every vertex v_i in G let us form a Boolean product of sums $(v_i + v_{i_1} + v_{i_2} + \cdots + v_{i_d})$, where $v_{i_1}, v_{i_2}, \ldots, v_{i_d}$ are the vertices adjacent to v_i, and d is the degree of v_i:

$$\theta = \prod (v_i + v_{i_1} + v_{i_2} + \cdots + v_{i_d}) \qquad \text{for all } v_i \text{ in } G.$$

$$\theta = (a + b)(b + c + d + e + a)(c + b +$$

$$(e + b + c + d + f + g)(f + e + g)($$ [redacted]

Since in Boolean arithmetic $(x + y)x = x$,

$$\theta = (a + b)(b + c + e)(b + d$$ [redacted]

$$= ae + be + bf + bg + acdf$$ [redacted]

Chromatic Polynomial