Chapter 10

Matrix Representations of Graphs

Chapter Goals

- Define adjacency matrix for a graph.
- Define the (reduced) incidence matrix for a graph.
- Present the Matrix-Tree Theorem for graphs.
- Define the (fundamental) circuit matrix for a graph.
- Define the (fundamental) edge cut matrix for a graph.
- Derive relationships among these matrices for a graph.
- Present an application in a switching network.
- Present an application in electrical network analysis.

10.1 Introduction

Although a pictorial representation of a graph is very convenient for visual study, other representations are better for computer processing. A matrix is a convenient and useful way of representing a graph to a computer. Matrices lend themselves easily to mechanical manipulations. Besides, numerous known results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. In many applications of graph theory, such as in electrical network analysis and operations research, matrices also turn out to be a natural way of expressing problems.

We first consider the two most frequently used matrix representations of a graph, the adjacency matrix in §10.2 and the incidence matrix in §10.3. In §10.4 we present the Matrix-Tree Theorem and its proof. We give an application to electrical circuit analysis in §10.5. In §10.6, §10.7, and §10.8 we cover the cycle matrix, fundamental cycle matrix, and edge cut matrix, respectively. We derive correspondences between some graph-theoretic properties and matrix properties in §10.9. In §10.10 we present path matrices. In view of the close tie between matrices and vector spaces, this chapter should be viewed as a continuation of Chapter 9. We assume some basic knowledge of matrices and matrix algebra. Any course in elementary linear algebra should suffice as a prerequisite. Throughout an $m \times n$ *matrix* $\mathsf{X} = [x_{ij}]$ denotes the matrix with m rows and n columns, where x_{ij} is the entry in row i and column j.

10.2 Adjacency Matrix

The *adjacency matrix* is probably the most frequently used matrix representation of a graph. It is always a square matrix and can therefore be manipulated in many more ways than nonsquare matrices.

Definition 10.1
Let $G = (V, E, \phi)$ be a graph on n vertices that are labeled by $V(G) = \{v_1, \ldots, v_n\}$. For each $i, j \in \{1, \ldots, n\}$ define the entry a_{ij} by

$$a_{ij} = |\{e \in E(G) : \phi(e) = \{v_i, v_j\}\}|.$$

That is, a_{ij} is the number of edges connecting v_i and v_j. The adjacency matrix *of G with respect to the labeling of $V(G)$ is defined as the $n \times n$ matrix*

$$\mathsf{A}(G) = [a_{ij}]_{i,j \in \{1,\ldots,n\}}.$$

We look at a simple example to make the definition clear.

☞ **Example 10.2**
Consider the simple graph G shown in Figure 10.1. Assume the vertices are labeled $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The corresponding adjacency matrix $\mathsf{A}(G)$

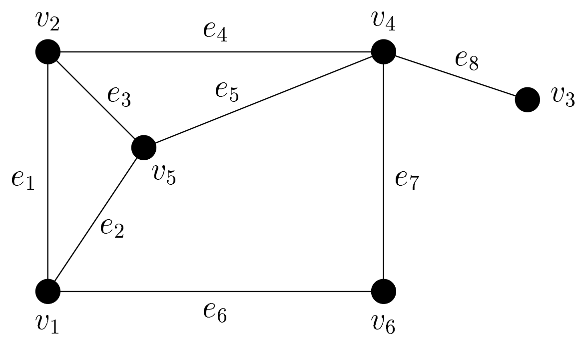

Figure 10.1: A simple graph G on six vertices has a 6×6 adjacency matrix.

is

$$\mathsf{A}(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Remark 10.3

- *We note that for any graph G and any listing of its vertices, the adjacency matrix $\mathsf{A}(G)$ is a symmetric matrix. That is, a_{ij} equals a_{ji} for all i and j. Equivalently, the matrix is equal to its transpose $\mathsf{A}(G)^t$ equals $\mathsf{A}(G)$. Moreover, any symmetric $n \times n$ matrix A with a_{ij} a nonnegative integer is an adjacency matrix for some graph G on n vertices.*

- *If G is a simple graph as in Example 10.2, then $\mathsf{A}(G)$ is a symmetric* binary matrix. *That is, $a_{ij} \in \{0, 1\}$ for all i and j. In addition, a_{ii} equals zero for each i since G has no loops.*

- *The number of edges in G is the sum of all of the entries a_{ij} where i is greater than or equal to j. Since $\mathsf{A}(G)$ is symmetric, the number of edges in G is also the sum of all a_{ij} where i is less than or equal to j. So, we have*

$$|E(G)| = \sum_{i \geq j} a_{ij} = \sum_{i \leq j} a_{ij}.$$

▷ *For each i the degree $d_G(v_i)$ is the sum of all entries in row (or column) number i in $\mathsf{A}(G)$, where we count the entry a_{ii} on the diagonal twice. That is,*

$$d_G(v_i) = \sum_{l=1}^{n} a_{il} + a_{ii} = \sum_{l=1}^{n} a_{li} + a_{ii}.$$

The notation $\mathsf{A}(G)$ for the adjacency matrix of G with respect to a given labeling $V(G) = \{v_1, \ldots, v_n\}$ of the vertices suggests that the matrix only depends on the graph itself. This is not quite true since we need a fixed labeling of the vertices. Suppose we have another labeling $V(G) = \{v'_1, \ldots, v'_n\}$, where $v'_i = v_{\sigma(i)}$ for some fixed permutation

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

If $\mathsf{A}(G)' = [a'_{ij}]_{i,j\in\{1,\ldots,n\}}$ is the adjacency matrix with respect to this labeling, then clearly

$$a'_{ij} = a_{\sigma(i)\sigma(j)}. \tag{10.1}$$

Let $\tilde{\sigma}(i)$ denote the column vector with a one in row $\sigma(i)$ and zero everywhere else. We can define the square *permutation matrix*

$$\mathsf{P}_\sigma = [\tilde{\sigma}(1) \mid \cdots \mid \tilde{\sigma}(n)].$$

Here the vertical bars are used to separate columns. This matrix is *orthogonal*, its inverse is given by its transpose, P_σ^{-1} equals P_σ^t. Moreover, if σ and ρ are two permutations on $\{1, \ldots, n\}$, then we have

$$\mathsf{P}_\sigma \mathsf{P}_\rho = \mathsf{P}_{\sigma\rho},$$

where $\sigma\rho$ denotes the usual composition of permutations (see Exercise 1). By Equation (10.1) we have

$$\mathsf{A}(G) = \mathsf{P}_\sigma^t \mathsf{A}(G) \mathsf{P}_\sigma = \mathsf{P}_\sigma^{-1} \mathsf{A}(G) \mathsf{P}_\sigma.$$

Definition 10.4
We say that two $n \times n$ matrices X and Y are orthogonally equivalent *if there is a permutation matrix P such that $\mathsf{Y} = \mathsf{P}^{-1}\mathsf{X}\mathsf{P}$.*

The proof of the following observation is left as Exercise 2.

Observation 10.5
Orthogonal equivalence among $n \times n$ matrices is an equivalence relation.

Two graphs G and G' being isomorphic means in particular that we have a bijective vertex map $f_1 : V(G) \to V(G')$ so there are equally many edges between u and v in G as there are between $f_1(u)$ and $f_1(u')$ in G'. If now

$$V(G) = \{u_1, \ldots, u_n\}$$

and

$$V(G') = \{u'_1, \ldots, u'_n\}$$

then there must be a permutation σ such that $f_1(u_i) = u'_{\sigma(i)}$ for each $i \in \{1, \ldots, n\}$. Hence, we have the following.

Observation 10.6
The following statements are equivalent:

- ▷ *The graphs G and G' are isomorphic.*
- ▷ *The adjacency matrices $\mathsf{A}(G)$ and $\mathsf{A}(G')$ are orthogonally equivalent with respect to any labeling of their vertices.*

We present an example to illustrate how various matrix manipulations relate to graph-theoretic properties.

☞ **Example 10.7**
Consider the simple graph G with its labeling as given in Example 10.2. By squaring the adjacency matrix $\mathsf{A}(G)$, we get the symmetric matrix

$$\mathsf{A}(G)^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

We now interpret the entries of this resulting square matrix. For each i and j denote by $a_{ij}^{(2)}$ the entry in row i and column j in $\mathsf{A}(G)^2$. In general, $a_{ij}^{(k)}$ denotes *the entry in row i and column j in $\mathsf{A}(G)^k$. By the rules of matrix multiplication, we have*

$$a_{ij}^{(2)} = \sum_{l=1}^{n} a_{il}a_{lj} = a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj}.$$

We observe that $a_{il}a_{lj}$ equals one only if v_i is connected to v_l and v_l is connected to v_j in G. Otherwise, $a_{il}a_{lj}$ equals zero. Therefore, $a_{ij}^{(2)}$ counts the number of walks of length two between v_i and v_j in G.

Similarly, we can consider the third power $\mathsf{A}(G)^3$ of the adjacency matrix of G.

$$\mathsf{A}(G)^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}.$$

For any square matrix X, we have $\mathsf{X}^3 = \mathsf{X} \cdot \mathsf{X}^2$. Hence, in particular for $\mathsf{A}(G)$, we have

$$a_{ij}^{(3)} = \sum_{l=1}^{n} a_{il}a_{lj}^{(2)} = a_{i1}a_{1j}^{(2)} + a_{i2}a_{2j}^{(2)} + \cdots + a_{in}a_{nj}^{(2)}.$$

We note that a_{il} equals one if and only if the vertex v_l is a neighbor of v_i. Otherwise, it is zero. For each neighbor v_l of v_i the entry $a_{kj}^{(2)}$ in $\mathsf{A}(G)$ counts the number of walks of length two between v_k and v_j. So, $a_{ij}^{(3)}$ counts the number of walks from v_i to v_j of length three.

For example, $a_{15}^{(3)} = a_{51}^{(3)} = 7$, indicating seven possible walks between v_1 and v_5 in G. Namely,

$$\begin{aligned} w_1 &= (v_1, e_1, v_2, e_1, v_1, e_2, v_5), \\ w_2 &= (v_1, e_2, v_5, e_2, v_1, e_2, v_5), \\ w_3 &= (v_1, e_6, v_6, e_6, v_1, e_2, v_5), \\ w_4 &= (v_1, e_2, v_5, e_3, v_2, e_3, v_5), \\ w_5 &= (v_1, e_6, v_6, e_7, v_4, e_5, v_5), \\ w_6 &= (v_1, e_2, v_5, e_5, v_4, e_5, v_5), \text{ and} \\ w_7 &= (v_1, e_1, v_2, e_4, v_4, e_5, v_5). \end{aligned}$$

Generalizing Example 10.7 we obtain the following theorem.

Theorem 10.8
Let G be a simple graph with vertex labeling $V(G) = \{u_1, \ldots, u_n\}$. Let k be a natural number greater than zero. The entry $a_{ij}^{(k)}$ is the number of distinct walks from u_i to u_j of length k in G.

PROOF: The theorem follows easily by induction on k using the same method as in Example 10.7 (see Exercise 6). □

A couple of corollaries follow from Theorem 10.8. Their proofs are left as Exercises 7 and 8, respectively.

Corollary 10.9
Let G be a connected graph with vertex labeling $V(G) = \{u_1, \ldots, u_n\}$. The distance between two distinct vertices u_i and u_j is the smallest natural number k for which the entry $a_{ij}^{(k)}$ in $\mathsf{A}(G)^k$ is nonzero.

Corollary 10.10
Let G be a graph on n vertices. Let

$$\mathsf{Y} = \mathsf{A}(G) + \mathsf{A}(G)^2 + \mathsf{A}(G)^3 + \cdots + \mathsf{A}(G)^{n-1}.$$

Then G is connected if and only if all entries of Y are nonzero.

10.3 Incidence Matrix

In this section we consider another important matrix representation of a given graph G, the *incidence matrix* of G. Unlike the adjacency matrix, the incidence matrix is not a square matrix in most cases. In addition to a labeling of the vertices, the incidence matrix also needs a labeling of the edges of G. Strictly speaking, the incidence matrix contains more information than the adjacency matrix since it distinguishes between multiple edges between two given vertices. The adjacency matrix does not.

Definition 10.11
Let $G = (V, E, \phi)$ be a graph with vertex labeling $V(G) = \{u_1, \ldots, u_n\}$ and edge labeling $E(G) = \{e_1, \ldots, e_m\}$. For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ define b_{ij} by

$$b_{ij} = |\{u_i\} \cap \phi(e_j)|.$$

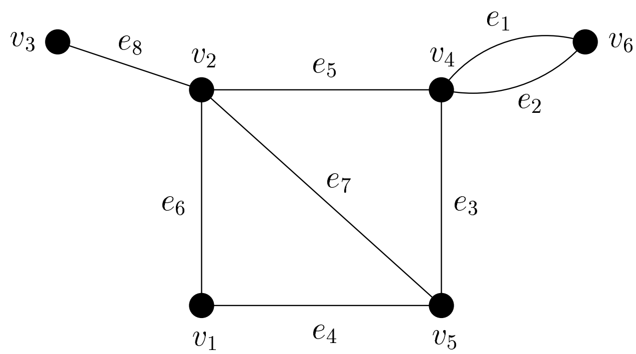

Figure 10.2: A graph G on six vertices and eight edges has a 6×8 incidence matrix.

That is, b_{ij} equals one if u_i is an end vertex of e_j and zero otherwise. The incidence matrix *of G with respect to these labelings of $V(G)$ and $E(G)$ is defined as the $n \times m$ binary matrix*

$$\mathsf{B}(G) = [b_{ij}]_{i \in \{1,\dots,n\}, j \in \{1,\dots,m\}}.$$

We present an example to make the definition clear.

Example 10.12

Consider the graph G shown in Figure 10.2. Assume the vertices are labeled $V(G) = \{v_1, \dots, v_6\}$ and the edges $E(G) = \{e_1, \dots, e_8\}$. According to this labeling, we have that the incidence matrix is given by

$$\mathsf{B}(G) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 10.13

- ▷ *Each column has either one or two 1's. Column j has exactly two 1's if the edge e_j has two distinct endvertices, and one 1 if e_j is a loop.*
- ▷ *The number of 1's in row i, where the 1's corresponding to a loop are counted twice, is equal to $d_G(u_i)$. In particular, if G has no loops, the number of 1's in row i is precisely $d_G(u_i)$.*

▷ *On the other hand, any $n \times m$ binary matrix such that each column has one or two 1's is an incidence matrix for a graph on n vertices and m edges.*

As with the adjacency matrix, the incidence matrix $\mathsf{B}(G)$ depends on the labelings of the vertices and the edges. Hence, two incidence matrices for a given graph G only differ by a permutation of the rows and columns. To be precise, let G and G' be graphs, where

$$\begin{aligned} V(G) &= \{u_1, \ldots, u_n\}, \\ E(G) &= \{e_1, \ldots, e_m\}, \\ V(G') &= \{u'_1, \ldots, u'_n\}, \text{ and} \\ E(G') &= \{e'_1, \ldots, e'_m\}. \end{aligned}$$

Assume $f = (f_1, f_2) : G \to G'$ is an isomorphism with

$$\begin{aligned} f_1(u_i) &= u'_{\sigma(i)}, \text{ and} \\ f_2(e_j) &= e'_{\rho(j)}, \end{aligned}$$

where σ and ρ are some permutations on $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively. Suppose P_σ and P_ρ are the corresponding $n \times n$ and $m \times m$ permutation matrices of σ and ρ, respectively. The incidence matrices satisfy

$$\mathsf{B}(G') = \mathsf{P}^t_\sigma \mathsf{B}(G) \mathsf{P}_\rho.$$

We summarize in the following observation.

Observation 10.14
Two graphs G and G' on n vertices and m edges are isomorphic if and only if there are $n \times n$ and $m \times m$ permutation matrices P *and* Q*, respectively such that*

$$\mathsf{B}(G') = \mathsf{P}^t \mathsf{B}(G) \mathsf{Q}.$$

As for adjacency matrices, this defines an equivalence relation among $n \times m$ matrices (see Exercise 11).

Recall that the row space and the column space of an $n \times m$ matrix over any field have the same dimension over the given field [17, Theorem 24, page 114].

In particular, if we view the incidence matrix as a matrix of the Galois field $\mathbb{Z}_2$, the row space and the column space of $\mathsf{B}(G)$ have the same dimension over $\mathbb{Z}_2$. This dimension is called the *rank* of the matrix. We leave it as Exercise 12 to show that if X and X′ are two $n \times m$ matrices such that $\mathsf{X}' = \mathsf{P}^t\mathsf{XQ}$, then they have the same rank. Therefore, the following definition is sensible.

Definition 10.15
The rank *of a graph G, denoted by* $\mathrm{rank}(G)$*, is the rank of its incidence matrix* $\mathsf{B}(G)$ *with respect to any labeling of $V(G)$ and $E(G)$.*

With this in mind, we can compute the rank of any connected graph G that has no loops.

Theorem 10.16
If G is a connected graph on n vertices without loops, then $\mathrm{rank}(G)$ *equals $n-1$.*

Proof: Since G has no loops, there are exactly two 1's in each column of $\mathsf{B}(G)$. Therefore, the sum of all row vectors of $\mathsf{B}(G)$ is the zero vector in $\mathbb{Z}_2^m$, provided that G has m edges. Hence, the dimension of the row space is $n-1$ or less. Therefore,

$$\mathrm{rank}(G) \leq n-1.$$

Assume we have n_1 less than $n-1$ rows that add up to the zero vector in $\mathbb{Z}_2^m$. The other $n_2 = n - n_1$ rows must also add up to the zero vector in $\mathbb{Z}_2^m$ since together they all add up to the zero vector. In this case we can permute the vertices of G in such a way that we first list the n_1 vertices that correspond to the rows adding up to the zero vector in $\mathbb{Z}_2^m$, and then the remaining n_2 ones that also add up to the zero vector. Since each column has exactly two 1's, each pair of 1's must either both be among the first n_1 entries of that column, or in the last n_2 ones. This means there is a permutation of the columns such that the first m_1 columns contain all the pairs of 1's among the first n_1 rows, and the last $m_2 = m - m_1$ columns contain all the pairs of 1's among the last n_2 rows. This means that the incidence matrix of G with respect to the corresponding labelings of $V(G)$ and $E(G)$ has the form

$$\mathsf{B}(G) = \begin{bmatrix} \mathsf{B}(G_1) & 0 \\ 0 & \mathsf{B}(G_2) \end{bmatrix},$$

where each $\mathsf{B}(G_1)$ is the incidence matrix for the subgraph of G induced by the first n_1 vertices, and $\mathsf{B}(G_2)$ is the incidence matrix induced by the last n_2

vertices of G. Since $\mathsf{B}(G)$ has this form, $G = G_1 \cup G_2$ must be disconnected (see Exercise 10). This contradicts our assumption. Since we are working in the Galois field $\mathbb{Z}_2$, we can conclude the only equation that the row vectors satisfy is that their sum is the zero vector in $\mathbb{Z}_2^m$. Hence, the rank of $\mathsf{B}(G)$ is at least $n-1$. This completes the proof of the theorem. □

As a corollary, we get the following (see Exercise 13).

Corollary 10.17
A graph G on n vertices that has k components has rank$(G) = n - k$.

If we remove any one row from the incidence matrix of a connected graph, the remaining $n-1$ by m submatrix is by Theorem 10.16 of rank $n-1$. In other words, the remaining $n-1$ row vectors sum up to the removed row. Thus we need only $n-1$ rows of an incidence matrix to specify the corresponding graph completely, for $n-1$ rows contain the same amount of information as the entire matrix.

Definition 10.18
Any $(n-1) \times m$ submatrix $\mathsf{B}_f(G)$ of an $n \times m$ incidence matrix $\mathsf{B}(G)$ of a connected graph G with no loops is called a reduced incidence matrix *of the graph G. The vertex corresponding to the deleted row of $\mathsf{B}(G)$ is called the* reference vertex *with respect to this reduced incidence matrix.*

Remark 10.19

▷ *Clearly, any vertex of a connected graph can be made the reference vertex.*

▷ *Since a tree is a connected loopless graph on n vertices and $n-1$ edges, its reduced incidence matrix is a square matrix of size and rank $n-1$.*

Recall that a square $n \times n$ matrix is *nonsingular* if it has rank n. Otherwise, it is *singular*.

Corollary 10.20
A reduced incidence matrix of a loopless connected graph G is nonsingular if and only if G is a tree.

Proof: A graph with n vertices and $n-1$ edges that is not a tree is disconnected. The rank of the incidence matrix of such a graph is less than $n-1$. Therefore, the $(n-1) \times (n-1)$ reduced incidence matrix of such a graph is singular. □

Let $\det(M)$ denote the determinant of matrix M. Let G' be a subgraph of a graph G, where G has n vertices and m edges. Let $\mathsf{B}(G')$ and $\mathsf{B}(G)$ be their incidence matrices, respectively. By permuting rows and columns, $\mathsf{B}(G')$ is a submatrix of $\mathsf{B}(G)$. There is a one-to-one correspondence between each $n \times k$ submatrix of the $n \times m$ incidence matrix $\mathsf{B}(G)$ and each subgraph of G having k edges. Submatrices of $\mathsf{B}(G)$ corresponding to special types of subgraphs such as cycles, spanning trees, or edge cuts in G exhibit special properties as the following theorem shows.

Theorem 10.21
Let G be a connected loopless graph on n vertices and m edges. Let $\mathsf{B}(G)$ be its $n \times m$ incidence matrix. An $(n-1) \times (n-1)$ submatrix B' of $\mathsf{B}(G)$ is nonsingular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree of G. In this case we also have that $\det(\mathsf{B}') = \pm 1$. This also holds for the matrix $\mathsf{B}_{-1}(G)$ where we have arbitrarily replaced one of the 1's in each column of $\mathsf{B}(G)$ by a -1.

PROOF: Every square submatrix of size $n-1$ in $\mathsf{B}(G)$ is the reduced incidence matrix of the same subgraph in G with $n-1$ edges, and vice versa. By Corollary 10.20 an $(n-1) \times (n-1)$ square submatrix B' of $\mathsf{B}(G)$ is nonsingular if and only if the corresponding subgraph is a tree. The tree in this case is a spanning tree because it contains $n-1$ edges of the graph G on n vertices. Since every tree has a leaf, there is a row in B' with exactly one 1 in it. Using induction on n and expanding the determinant along that row completes the proof. □

10.4 The Matrix-Tree Theorem

In this section we present an important interpretation of the adjacency matrix of a simple graph G on n vertices. We assume that G has m edges unless otherwise stated.

As mentioned in the previous section, when a graph G has no loops, each row of the incidence matrix $\mathsf{B}(G)$ has precisely two 1's. If in addition G has no multiple edges then the adjacency matrix $\mathsf{A}(G)$ is also a binary matrix. Hence, given a simple graph it is natural to ask about the relationship between the incidence matrix and the adjacency matrix. We begin with some definitions.

Definition 10.22
Let G be a graph and $V(G) = \{u_1, \ldots, u_n\}$ be a labeling of its vertex set. The diagonal matrix $\mathsf{D}(G)$, where the i-th diagonal entry is the degree of u_i is called the degree matrix *of G with respect to the labeling $V(G)$.*

Viewing $\mathsf{B}_{-1}(G)$ from Theorem 10.21 as a row matrix

$$\mathsf{B}_{-1}(G) = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix},$$

we see that the entry in row i and column j in the $n \times n$ matrix $\mathsf{B}_{-1}(G) \cdot \mathsf{B}_{-1}(G)^t$ is given by the dot product $\tilde{b}_i \cdot \tilde{b}_j$. In particular, we have that the i-th diagonal entry equals $d_G(u_i)$ for each vertex u_i of G. Also, if i does not equal j, then $\tilde{b}_i \cdot \tilde{b}_j$ equals zero if the vertices u_i and u_j are not connected. However, $\tilde{b}_i \cdot \tilde{b}_j = 1 \cdot (-1) = -1$ if they are connected. From this we have the following relation between the adjacency matrix $\mathsf{A}(G)$ and the incidence matrix $\mathsf{B}(G)$ of G.

Observation 10.23
For a simple graph G we have that

$$\mathsf{A}(G) + \mathsf{B}_{-1}(G) \cdot \mathsf{B}_{-1}(G)^t = \mathsf{D}(G).$$

The matrix $\mathsf{D}(G) - \mathsf{A}(G)$ from Observation 10.23 has some additional interesting properties. We need the *Binet-Cauchy Theorem* to prove some of these results. This theorem is usually not covered in the standard curriculum of linear algebra so we record it here.

Theorem 10.24 (Binet-Cauchy)
Let m and n be nonnegative integers, where m is greater than or equal to n. Let X be $n \times m$ matrix and Y an $m \times n$ matrix, where their entries are elements of a given ring. For each $S = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$, let X_S (Y_S) be the $n \times n$ square matrix obtained by choosing the columns (respectively, rows) number i_1 through i_n from X (respectively, Y). In this case we have

$$\det(\mathsf{XY}) = \sum_{S \subseteq \{1,\ldots,m\}} \det(\mathsf{X}_S)\det(\mathsf{Y}_S),$$

where the sum is taken over all the $\binom{m}{n}$ subsets S of $\{1, \ldots, m\}$ that contain precisely n elements.

Before proving this theorem, we illustrate it with an example.

☞ **Example 10.25**

Let

$$\mathsf{X} = \begin{bmatrix} 4 & -3 & -2 \\ 2 & -1 & 0 \end{bmatrix} \textit{ and } \mathsf{Y} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 3 & -2 \end{bmatrix}.$$

In this case n equals two, m equals three, and $\binom{3}{2}$ equals three. So, we have that $\det(\mathsf{XY})$ is the following sum:

$$\begin{aligned} \det(\mathsf{XY}) &= \det \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & -1 \\ -2 & -0 \end{bmatrix} \\ &\quad + \det \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \\ &\quad + \det \begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix} \cdot \det \begin{bmatrix} -2 & 0 \\ 3 & -2 \end{bmatrix} \\ &= 2 \cdot (-2) + 4 \cdot 1 + (-2) \cdot 4 \\ &= -8. \end{aligned}$$

Note that XY *is*

$$\begin{bmatrix} 4 & 0 \\ 4 & -2 \end{bmatrix}$$

and

$$\det \begin{bmatrix} 4 & 0 \\ 4 & -2 \end{bmatrix} = -8.$$

To prove the Binet-Cauchy Theorem, we present the following technical lemma. It can be viewed as a generalization of the Binet-Cauchy Theorem. The statement of the result allows for a short proof.

Lemma 10.26

Let X *and* Y *be $n \times m$ and $m \times n$ matrices, respectively. Let $T \subseteq \{1, \ldots, m\}$ and let P_T be the $m \times m$ diagonal matrix whose i-th diagonal entry is equal to one if and only if $i \in T$. Let* M *be the $(m+n) \times (m+n)$ matrix defined by*

$$\mathsf{M} = \begin{bmatrix} \mathsf{P}_T & \mathsf{Y} \\ \mathsf{X} & 0 \end{bmatrix}.$$

Let $T' = \{1, \dots, m\} \setminus T$. If $|T|$ is less than $m - n$, then $\det(\mathsf{M})$ equals zero. Otherwise, using the notation from Theorem 10.24, we have that

$$\det(\mathsf{M}) = \sum_{T' \subseteq S \subseteq \{1,\dots,m\}} \det(\mathsf{X}_S) \det(\mathsf{Y}_S),$$

where the sum is taken over all

$$\binom{|T|}{n - |T'|}$$

subsets S that contain T' and have exactly n elements.

PROOF: Clearly, if $|T|$ is less than $m - n$, then the rows (or columns) corresponding to T' are linearly dependent and hence the determinant is zero.

Otherwise, we expand the determinant of M by any row (or column) numbered $i \in T$ and we get

$$\det(\mathsf{M}) = \det \begin{bmatrix} \mathsf{P}_{T;i} & \mathsf{Y}_i \\ \mathsf{X}_i & 0 \end{bmatrix} + \det \begin{bmatrix} \mathsf{P}_T^0 & \mathsf{Y} \\ \mathsf{X} & 0 \end{bmatrix},$$

Here the first matrix is an $(m + n - 1) \times (m + n - 1)$ one, where $\mathsf{P}_{T;i}$ is the $(m-1)\times(m-1)$ matrix obtained from P_T by removing row and column i, and X_i (Y_i) is obtained from X (respectively, Y) by removing column (respectively row) i from X (respectively, Y). Likewise, the second matrix is the $(m+n) \times (m+n)$ one, where P_T^0 is obtained from P_T by changing the i-th diagonal entry from one to zero. The lemma now follows by induction on $m + n + |T|$. □

Using Lemma 10.26, we can now prove Theorem 10.24.

PROOF:(Theorem 10.24) We note that

$$\begin{bmatrix} \mathsf{I}_m & 0 \\ -\mathsf{X} & \mathsf{I}_n \end{bmatrix} \cdot \begin{bmatrix} -\mathsf{I}_m & \mathsf{Y} \\ \mathsf{X} & 0 \end{bmatrix} = \begin{bmatrix} -\mathsf{I}_m & \mathsf{Y} \\ 0 & \mathsf{XY} \end{bmatrix},$$

where I_α is the $\alpha \times \alpha$ identity matrix. From this we get that

$$\det \begin{bmatrix} -\mathsf{I}_m & \mathsf{Y} \\ \mathsf{X} & 0 \end{bmatrix} = \det \begin{bmatrix} -\mathsf{I}_m & \mathsf{Y} \\ 0 & \mathsf{XY} \end{bmatrix},$$

and hence

$$\det(\mathsf{XY}) = -\det \begin{bmatrix} -\mathsf{I}_m & \mathsf{Y} \\ \mathsf{X} & 0 \end{bmatrix}. \quad (10.2)$$

Applying Lemma 10.26 on the right hand side of Equation (10.2) completes the proof. □

At this point we can present the Matrix-Tree Theorem that describes an important property of the $n \times n$ matrix $\mathsf{D}(G) - \mathsf{A}(G)$ for any simple graph G on n vertices.

Let M be an $n \times n$ matrix. Let B be the matrix obtained from M by omitting the i-th row and j-th column. The *cofactor* of element a_{ij} of M is equal to

$$(-1)^{i+j} \det(B).$$

Theorem 10.27 (Matrix-Tree)
If G is a simple graph, then all of the cofactors of the matrix $\mathsf{D}(G) - \mathsf{A}(G)$ are equal to $\tau(G)$, the number of spanning trees of G.

Proof: By Observation 10.23 we have that

$$\mathsf{D}(G) - \mathsf{A}(G) = \mathsf{B}_{-1}(G) \cdot \mathsf{B}_{-1}(G)^t.$$

From this we note that the (i,i)-th cofactor of $\mathsf{D}(G) - \mathsf{A}(G)$ is obtained by the matrix product $\mathsf{B}_{-1;i}(G) \cdot \mathsf{B}_{-1;i}(G)^t$, where $\mathsf{B}_{-1;i}(G)$ is obtained from $\mathsf{B}_{-1}(G)$ by removing the i-th row. By the Binet-Cauchy Theorem 10.24 $\det(\mathsf{B}_{-1;i}(G) \cdot \mathsf{B}_{-1;i}(G)^t)$ equals the sum of all summands $\mathsf{B}' \cdot \mathsf{B}'^t$, where B' is a nonsingular $(n-1) \times (n-1)$ submatrix of $\mathsf{B}_{-1}(G)$. By Theorem 10.21 there are precisely $\tau(G)$ such summands and each summand is equal to $(\pm 1)^2 = 1$. We therefore have that each (i,i)-th cofactor of $\mathsf{D}(G) - \mathsf{A}(G)$ equals to $\tau(G)$.

To show every (i,j)-th cofactor is equal to $\tau(G)$, we first note that the sum of all the rows in $\mathsf{X} = \mathsf{D}(G) - \mathsf{A}(G)$ is the zero vector. Consider now the submatrices $\mathsf{X}_{i,j}$ and $\mathsf{X}_{j,j}$ we obtain from X by removing row i and column j, and row j and column j, respectively. The corresponding cofactors are

$$\begin{aligned} C_{i,j} &= (-1)^{i+j} \det(\mathsf{X}_{i,j}) \text{ and} \\ C_{j,j} &= (-1)^{j+j} \det(\mathsf{X}_{j,j}) = \det(\mathsf{X}_{j,j}). \end{aligned}$$

Recall that a determinant remains unaltered when we add a multiple of one row to a given row. Hence, adding rows $1, \ldots, i-1, i+1, \ldots, n$ to the i-th row in $\mathsf{X}_{i,j}$ will not change $C_{i,j}$. Since the rows in X sum up to zero, the i-th row in $\mathsf{X}_{i,j}$, after this alteration, equals minus the j-th row of $\mathsf{X}_{j,j}$. Finally moving this altered i-th row to the j-th place, by means of $|j-i|-1$ interchanges of rows, yields the matrix $X_{j,j}$, where row i is multiplied by -1. Since each interchange reverses the sign of the determinant, we finally have that

$$C_{i,j} = (-1)^{i+j+|j-i|-1}(-C_{j,j}) = C_{j,j}.$$

This completes the proof of the theorem. □

We call the equation implied by Theorem 10.27 the *Matrix-Tree Formula*.

This gives us a second method to compute the number of spanning trees in any simple graph. Recall, the first was given in Theorem 5.4. Moreover, the Matrix-Tree Formula is mechanical and well suited when graphs have been implemented as adjacency or incidence matrices for computer computations.

The following example gives us an alternative way to prove Cayley's Theorem 5.7.

☞ **Example 10.28**
For n greater than two we have for the complete graph K_n on n vertices that

$$\mathsf{D}(K_n) - \mathsf{A}(K_n) = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

By the Matrix-Tree Theorem 10.27, we have that $\tau(K_n)$ equals to any cofactor of the preceding matrix. Considering the $(1,1)$ cofactor $C_{1,1}$ and then subtracting the first row from all the remaining rows, we get that $\tau(K_n)$ is given by the following $(n-1)\times(n-1)$ determinant:

$$\begin{aligned} \tau(K_n) &= \det \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix} \\ &= \det \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -n & & & \\ \vdots & & n \cdot \mathsf{I}_{n-2} & \\ -n & & & \end{bmatrix}, \end{aligned}$$

where I_{n-2} *denotes the* $(n-2) \times (n-2)$ *identity matrix. Adding columns* $2, \ldots, n-1$ *to the first column, we finally get*

$$\tau(K_n) = \det \begin{bmatrix} 1 & -1 & \cdots & -1 \\ 0 & & & \\ \vdots & & n \cdot \mathsf{I}_{n-2} & \\ 0 & & & \end{bmatrix} = n^{n-2}.$$

This completes the proof of Cayley's Theorem.

10.5 An Application in Electrical Circuits

In this section we discuss an application for a given electrical network system, where we need to find the total resistance between two points in the network system. We do not assume any prior knowledge in electrical physics although some knowledge is helpful to really appreciate what this section has to offer. We give a "crash"-overview of the underlying physical knowledge needed to continue.

We consider networks of the type that are displayed in Figure 10.3. Our electrical circuits are provided with a power supply having a given potential between two nodes u and v that we denote by V_{uv}, or simply V when there is no danger of ambiguity. We assume the electrical devices are connected to a power supply in an arbitrary fashion with combined resistance R. In this case the total electrical current I flowing from one node to the other is given by the formula

$$I = \frac{V_{uv}}{R}.$$

Hence, if we can compute the total resistance R, we can calculate the total flow I of electrical current.

The problem we are concerned with is to compute the total combined resistance of the network in question. The two basic rules to calculate the total resistance between two nodes u and v in a given network are as follows. If we have electrical devices that are connected in a series as shown in Figure 10.4, then the total resistance R_t is given by the sum of all the resistors

$$R_t = R_1 + R_2 + \cdots + R_n. \tag{10.3}$$

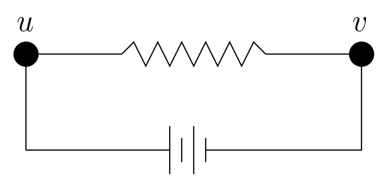

Figure 10.3: An electrical circuit with a given potential and a system of resistors, shown as one combined resistor.

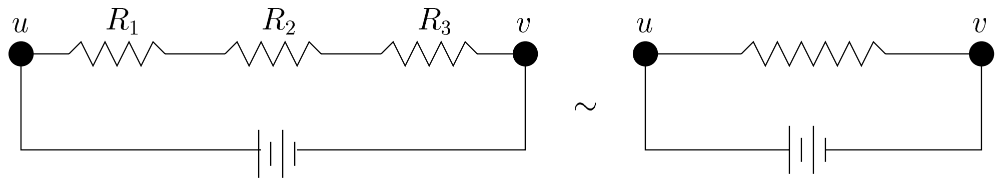

Figure 10.4: The resistors are connected in series.

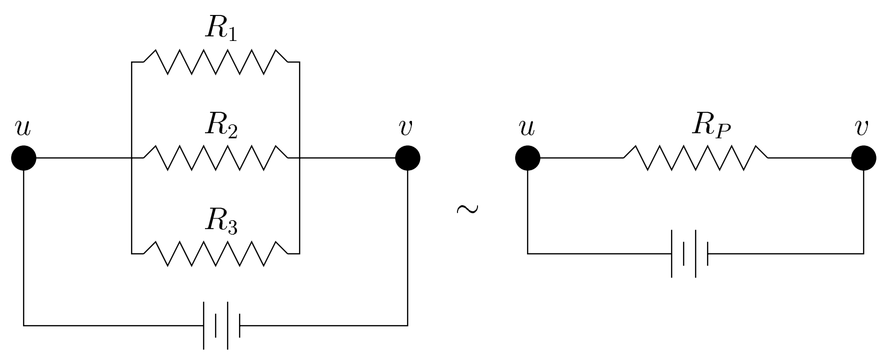

Figure 10.5: The resistors are connected in parallel.

If however the resistors are connected in parallel as shown in Figure 10.5, then the total resistance R_t satisfies the formula

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}. \tag{10.4}$$

Any introductory book on physics explains why Equations 10.3 and 10.4 hold. There are many other physical phenomena that behave in a similar fashion, but for simplicity's sake we shall restrict ourselves to networks involving only a potential, electrical current, and resistance.

The salient point to notice is that the total resistance of many resistor network

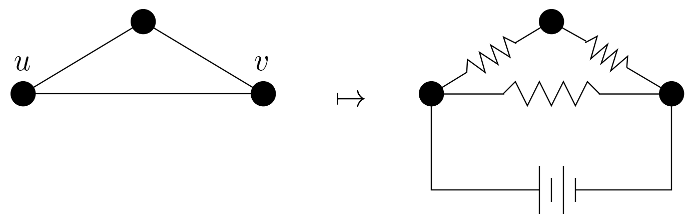

Figure 10.6: A graph and the corresponding electrical network.

systems can be calculated by repeated use of the above two formulas. However, there are countless other systems where they fail to be of any use without some geometrical insight or physical tricks. The proof of the following theorem can be found in [29, Proposition 2.3]. It can be used to compute the total resistance in a completely mechanical way between two nodes u and v in many more cases without any prior knowledge in physics.

Theorem 10.29

Let G be a simple graph. Let u and v be two neighboring vertices of G. Assume that each edge in G corresponds to the resistance of one ohm (1 Ω). In this case the total resistance of the network between u and v is given by

$$R_t = \frac{\tau_{uv}(G)}{\tau(G)},$$

where $\tau_{uv}(G)$ denotes the number of spanning trees that include the edge $\{u, v\}$.

➤ **Note 10.30**

As seen from the proof of Theorem 5.4, the number $\tau(G \cdot e)$ is precisely the number of spanning trees of G that contain the edge e. Hence from Theorem 5.4, we can write R_t as

$$R_t = \frac{\tau(G \cdot \{u, v\})}{\tau(G)} = \frac{\tau(G \cdot \{u, v\})}{\tau(G \cdot \{u, v\}) + \tau(G \setminus \{u, v\})}.$$

☞ **Example 10.31**

Consider the graph G shown on the left in Figure 10.6 and the corresponding electrical network, where each edge represents a resistance of one ohm.

Here we can easily compute the total resistance between u and v using the classical Formulas (10.3) and (10.4). By these formulas the total resistance R_t satisfies

$$\frac{1}{R_t} = \frac{1}{1+1} + \frac{1}{1},$$

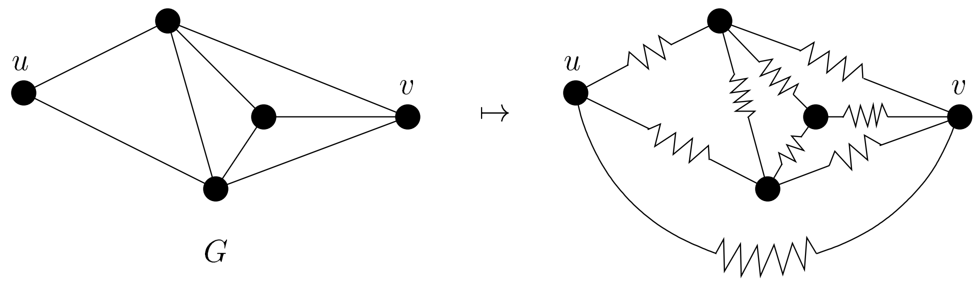

Figure 10.7: A graph G and a corresponding electrical network.

and hence we get $R_t = 2/3\ \Omega$.

Since the vertices u and v are neighbors in G, we get by Theorem 10.29 that

$$R_t = \frac{\tau_{uv}(G)}{\tau(G)} = \frac{2}{3},$$

in complete agreement with the classical formulas.

Theorem 10.29 gives the resistance between two points provided that the corresponding vertices in the graph are neighbors. If the vertices u and v are not neighbors, then we need in addition to Theorem 10.29, the Formula (10.4) as the following example shows.

☞ **Example 10.32**

Consider the graph G shown on the left in Figure 10.7. The vertices u and v are not neighbors. A direct application of the classical formulas will not be sufficient.

Adding the edge $e = \{u, v\}$, we obtain the graph G' and the corresponding electrical network displayed on the right in Figure 10.7. Let G'' be the graph we obtain from G by collapsing u and v into one vertex. Now, if R_t is the total resistance of the network corresponding to G and R_t' is the total resistance corresponding to G', we have by Formula (10.4) that

$$\frac{1}{R_t'} = \frac{1}{R_t} + \frac{1}{1}.$$

Since $R_t' = \tau_{uv}(G')/\tau(G')$ by Theorem 10.29, we have by Theorem 5.4 that

$$R_t = \frac{\tau_{uv}(G')}{\tau(G') - \tau_{uv}(G')} = \frac{\tau_{uv}(G')}{\tau(G' \setminus \{u, v\})} = \frac{\tau(G'')}{\tau(G)}.$$

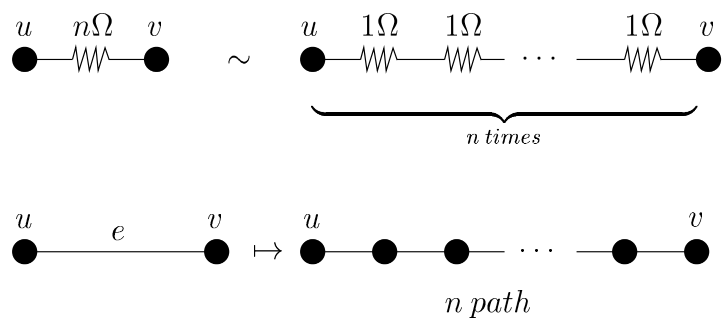

Figure 10.8: An a ohm resistor corresponds to a subdivided edge.

We observe that Theorem 10.29 reduces the tricky problem of determining the total resistance between two given points of a large class of electrical networks to that of computing the number of spanning trees of corresponding graphs. The Matrix-Tree Theorem 10.27 reduces the problem of computing the number of spanning trees of graphs to the problem of evaluating a determinants. Since there are many efficient ways of evaluating determinants of square matrices, we have a mechanical and efficient way of computing the total resistance of general electrical networks, where each edge of the corresponding graphs represents one ohm.

Now, what about general networks where the edges represent different resistance? In this case there is a partial algorithmic solution. Suppose we have an electrical network of resistors where each resistor can represent any a ohms, where a is a positive integer. Here looking at the corresponding graph for the network, we replace each edge representing a ohms with a simple path of length n (see Figure 10.8). By this subdivision we have that all edges represent a resistance of one ohm.

Suppose now that we have an electrical network of resistors where each resistor can represent any $1/b\ \Omega$, where b is a positive integer. Here looking at the corresponding graph for the network, we replace each edge representing $1/b\ \Omega$ with n copies of parallel edges (see Figure 10.9).

The same graph would be obtained by adding $b-1$ parallel edges to the one edge representing the $1/b\ \Omega$ resistor. By this procedure of generating a multigraph from our original graph, we have that all edges in this multigraph represent a resistance of one ohm.

We now combine these two methods. In a network of resistors, where each

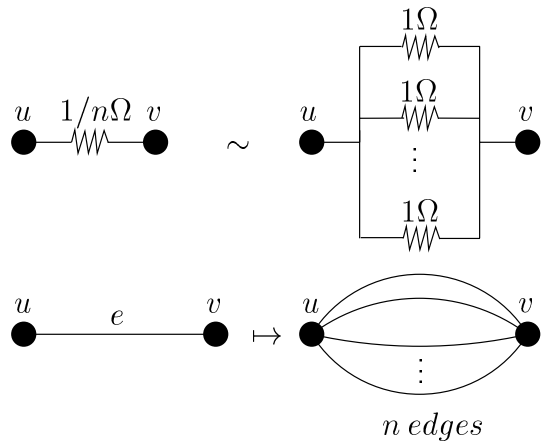

Figure 10.9: An $1/b\ \Omega$ resistor corresponds to n parallel edges.

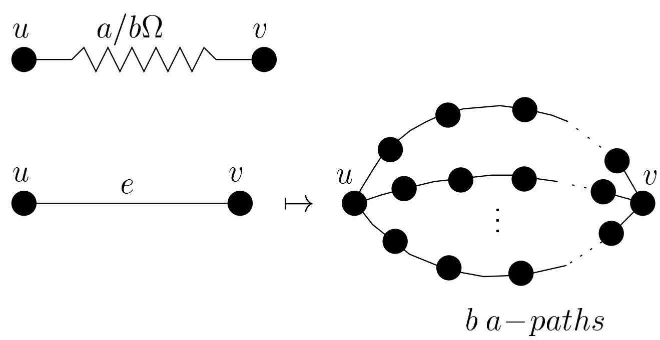

Figure 10.10: An $a/b\ \Omega$ resistor corresponds to a b parallel a paths.

resistor can represent any rational resistance $a/b\ \Omega$, replacing the corresponding edge by b parallel a-paths (see Figure 10.10), using the previous procedure yields a graph where each edge corresponds to one ohm. We conclude by the following observation.

Observation 10.33

The total resistance of an electrical network of rational resistors can be computed by applying the Matrix-Tree Theorem to the "altered" corresponding graph.

➤ **Note 10.34**

The procedure described for computing the total resistance of a given electrical network of resistors does not provide us with an exact *solution if some resistors have irrational resistance x ohms, where $x \in \mathbb{R} \setminus \mathbb{Q}$. However, from a practical* *point of view, our method does provide a computational method for any real-world network since every resistor is given by a floating number of finitely many decimal (or binary) digits that does indeed represent a rational number.*

10.6 Cycle Matrix

So far, we have discussed two matrices associated with a graph G, the adjacency matrix $\mathsf{A}(G)$ and the incidence matrix $\mathsf{B}(G)$. In this section we consider yet another matrix associated with a given graph, namely the *cycle matrix* of G.

Definition 10.35
Let G be a graph with m edges and q cycles, where the edges and cycles are labeled as $E(G) = \{e_1, \ldots, e_m\}$ and $\mathcal{C}(G) = \{\gamma_1, \ldots, \gamma_q\}$, respectively. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, q\}$ define c_{ij} by

$$c_{ij} = |E(\gamma_i) \cap \{e_j\}|.$$

That is, c_{ij} equals one if cycle γ_i includes edge e_j and is zero otherwise. The cycle matrix *of G with respect to the labelings of $E(G)$ and $\mathcal{C}(G)$ is the $q \times m$ binary matrix*

$$\mathsf{C}(G) = [c_{ij}]_{i\in\{1,\ldots,q\},j\in\{1,\ldots,m\}}.$$

The next example illustrates this definition.

☞ **Example 10.36**
Consider the graph G shown in Figure 10.2. It has four cycles

$$\begin{aligned} \gamma_1 &= \{e_1, e_2\}, \\ \gamma_2 &= \{e_3, e_5, e_7\}, \\ \gamma_3 &= \{e_4, e_6, e_7\}, \text{ and} \\ \gamma_4 &= \{e_3, e_4, e_6, e_5\}. \end{aligned}$$

Therefore, the cycle matrix $\mathsf{C}(G)$ is given by

$$\mathsf{C}(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Remark 10.37

▷ *A column of zeros in a cycle matrix* $\mathsf{C}(G)$ *of a graph* G *corresponds to an edge that does not belong to any cycle.*

▷ *Each row of* $\mathsf{C}(G)$ *is a vector representing a cycle.*

▷ *A row corresponding to a loop has a single one in it.*

▷ *The number of ones in a row is equal to the number of edges in the corresponding cycle.*

As with the incidence matrix $\mathsf{B}(G)$, the cycle matrix $\mathsf{C}(G)$ depends on the labelings of the edges and the cycles. A permutation of rows or columns in a cycle matrix simply corresponds to relabeling the cycles and edges. However, unlike incidence matrices, two cycle matrices that differ by permutations of the rows and columns, do not necessarily represent the same graph. They represent graphs that are 2-isomorphic, a la Theorem 6.47. We summarize in the following observation that is analogous to Observation 10.14.

Observation 10.38
Two graphs G *and* G' *on* m *edges having* q *cycles are 2-isomorphic if and only if there are* $q \times q$ *and* $m \times m$ *permutation matrices* P *and* Q*, respectively such that*

$$\mathsf{C}(G') = \mathsf{P}^t\mathsf{C}(G)\mathsf{Q}.$$

The next example illustrates Observation 10.38.

Example 10.39
The two graphs G *and* G'' *shown in Figure 6.13 have identical cycle matrices since they are 2-isomorphic. However, the graph are not isomorphic.*

Consider a separable (or disconnected) graph G with two blocks (or components) G_1 and G_2. Since each cycle of G must be either entirely in G_1 or G_2, we can label the edges and the cycles in such a way that the cycle matrix $\mathsf{C}(G)$ can be written in block diagonal form

$$\mathsf{C}(G) = \begin{bmatrix} \mathsf{C}(G_1) & 0 \\ 0 & \mathsf{C}(G_2) \end{bmatrix}, \tag{10.5}$$

where $\mathsf{C}(G_1)$ is the incidence matrix for the subgraph G_1 of G and $\mathsf{C}(G_2)$ is the incidence matrix for G_2. This can of course be generalized to more than two blocks (or components).

An important theorem relating the incidence matrix and the cycle matrix of a graph is the following.

Theorem 10.40
Let G be a graph. For a given labeling of the vertices, edges, and cycles of G, every row of $\mathsf{C}(G)$ is orthogonal to every row of $\mathsf{B}(G)$ over the Galois field of $\mathbb{Z}_2$. That is to say, computing in $\mathbb{Z}_2$, we have that

$$\mathsf{B}(G)\mathsf{C}(G)^t = \mathsf{C}(G)\mathsf{B}(G)^t = 0.$$

PROOF: We first note that for a given vertex u_i and a given cycle γ_j of G, either u_i is in γ_j or not.

If u_i is not in γ_j, then no edge incident to u_i is contained in γ_j. In this case the dot product of row i of $\mathsf{B}(G)$ and row j of $\mathsf{C}(G)$ is zero in $\mathbb{Z}$, and hence also in $\mathbb{Z}_2$.

If u_i is in γ_j, then there are exactly two edges in γ_j incident to u_i. In this case the dot product of row i of $\mathsf{B}(G)$ and row j of $\mathsf{C}(G)$ is exactly two in $\mathbb{Z}$, and hence zero in $\mathbb{Z}_2$. □

The next example illustrates Theorem 10.40.

☞ **Example 10.41**
Consider the graph shown in Figure 10.2. Using the labelings of $V(G)$, $E(G)$, and $\mathcal{C}(G)$ that were given in Examples 10.12 and 10.36, we get by computing in the Galois field $\mathbb{Z}_2$ that

$$\begin{aligned}\mathsf{B}(G)\mathsf{C}(G)^t &= \begin{bmatrix} 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&1\\ 1&1&1&0&1&0&0&0\\ 0&0&1&1&0&0&1&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0\\ 1&0&0&0\\ 0&1&0&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix}\\ &= \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}.\end{aligned}$$

10.7 Fundamental Cycle Matrix

As we saw in Chapters 4 and 5, any set of fundamental cycles with respect to any spanning tree in a connected graph forms an independent set of cycles in a graph. All other cycles can be formed by symmetric difference of the fundamental cycles. In other words, each cycle vector is a linear combination of the cycle vectors of the fundamental cycles over the Galois field $\mathbb{Z}_2$. Also, no fundamental cycle vector is a linear combination of other fundamental cycle vectors. So, any set of fundamental cycles form a set of basic cycles for the graph. Hence, in a cycle matrix, if we retain only those rows that correspond to a set of basic cycles and remove all other rows, we would not lose any information. The remaining rows can be reconstituted from the rows corresponding to the set of fundamental cycles.

☞ **Example 10.42**
Consider again the graph G shown in Figure 10.2. As we saw in Example 10.36, its cycle matrix $\mathsf{C}(G)$ has four rows each corresponding to one of its cycles. We note however that the fourth row is the sum of the second and third row of $\mathsf{C}(G)$. The reason is because $\gamma_4 = \gamma_2 \vartriangle \gamma_3$.

Recall from Lemma 8.14 that all collections of basic cycles in a graph on n vertices and m edges has precisely $m-n+1$ cycles. This motivates the following definition.

Definition 10.43
Let G be a graph on n vertices and m edges. Let $\mathsf{C}(G)$ be the cycle matrix of G with respect to some labeling of the edges and cycles.

Any $(m-n+1)\times m$ submatrix of $\mathsf{C}(G)$, where the $m-n+1$ rows correspond to a set of fundamental cycles with respect to a spanning tree of G is called a fundamental cycle matrix *of G and is denoted by $\mathsf{C}_f(G)$.*

The next example illustrates this definition.

☞ **Example 10.44**
Consider the graph G shown in Figure 10.11 where a spanning tree is indicated by thick edges. In this case, the fundamental cycle matrix of G with respect to the spanning tree and the labeling

$$E(G) = \{e_2, e_3, e_6, e_1, e_4, e_5, e_7\},$$

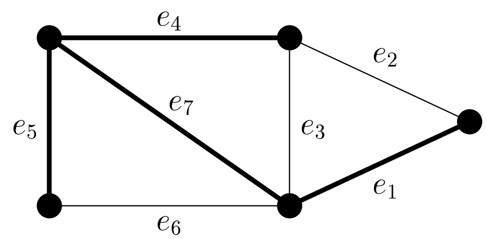

Figure 10.11: A graph with the edges labeled and a spanning tree shown in thick edges.

indicated in the figure, is given by

$$\mathsf{C}_f(G) = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right].$$

Note, in the matrix e_2 *corresponds to column one,* e_3 *to column two, and so on.*

As for the other matrices associated with a graph G, any permutation of rows and columns, corresponds to a relabeling of the cycles and edges, respectively.

For a graph G on n vertices and m edges with a fixed spanning tree T, we can label the m edges e_i and fundamental cycles γ_j in the following way:

1. The first $m - n + 1$ edges are the chords of G with respect to T.
2. For each $i \in \{1, \ldots, m - n + 1\}$, the chord e_i is contained in the fundamental cycle γ_i.

This is indeed the case in Figure 10.11. In this way we see that the fundamental cycle matrix has the form

$$\mathsf{C}_f(G) = [\mathsf{I}_\mu \mid \mathsf{C}_b(G)], \tag{10.6}$$

where $\mu = m - n + 1$ and $\mathsf{C}_b(G)$ is the $m \times (n - 1)$ matrix corresponding to the remaining branches of the spanning tree T. From Equation (10.6) it is clear that the rank of $\mathsf{C}_f(G)$ over $\mathbb{Z}_2$ is $\mu = m - n + 1$.

Since $\mathsf{C}_f(G)$ is a submatrix of the cycle matrix $\mathsf{C}_f(G)$, with same number of columns but fewer rows, we have that the rank of $\mathsf{C}(G)$ is at least $\mu = m-n+1$. Since every other row of $\mathsf{C}(G)$ is a linear combination of the rows of $\mathsf{C}_f(G)$ over $\mathbb{Z}_2$, we have the following fact.

Theorem 10.45
If a graph G has n vertices and m edges then

$$\text{rank}(\mathsf{C}(G)) = m - n + 1.$$

In fact, Theorem 10.45 is also a consequence of Corollary 9.55.

Note that in discussing spanning trees of a graph G it is necessary to assume that G is connected. In the case of a disconnected graph, we would have to consider a spanning forest and fundamental cycles with respect to this forest. By considering each component, we have the following corollary.

Corollary 10.46
If G is a disconnected graph having k components, n vertices, and m edges, then the rank of $\mathsf{C}_f(G)$ is $m - n + k$.

We now consider an example to apply our cycle matrices to a problem in an electrical switching network system. We assume the bare minimum knowledge in electrical physics.

☞ **Example 10.47**
Suppose you are given a black box that contains a switching network consisting of eight switches A, B, C, D, E, F, G, and H. The switches can be turned on or off from outside. You are asked to determine how the switches are connected inside without opening the black box.

One way to find the answer is to connect a lamp at the available terminals in series with a battery and an additional switch K, as shown in Figure 10.12, and then find out which of the various combinations light up the lamp. Note we need to have switch K on in every combination that turns on the lamp. In this experiment, suppose the combinations that turn on the lamp are the following eight:

$$\{A,B,F,H,K\}, \quad \{A,B,G,K\}, \quad \{A,E,F,G,K\}, \quad \{A,E,H,K\},$$
$$\{B,C,E,H,K\}, \quad \{C,F,H,K\}, \quad \{C,G,K\}, \text{ and } \{D,K\}. \tag{10.7}$$

To find out how the switches are connected we proceed as follows: Consider the switching network as a graph G whose edges represent switches. We can assume that the graph is connected, and has no loops. Since a lit lamp implies the formation of a cycle, we can regard the preceding list as a partial list of cycles in the corresponding graph. We start by listing edges and the eight cycles from List (10.7) by

$$\begin{aligned} E(G) &= \{A, B, C, D, E, F, G, H, K\} \text{ and} \\ \mathcal{C}(G) &= \{\gamma_1, \ldots, \gamma_8\}, \end{aligned}$$

where the order of the cycles is given in List (10.7).

$$\mathsf{C}(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, to simplify the matrix, we remove the obviously redundant cycles. Observe that the following symmetric differences of cycles give rise to other cycles.

$$\begin{aligned} \{A, B, G, K\} \vartriangle \{C, F, H, K\} \vartriangle \{C, G, K\} &= \{A, B, F, H, K\}, \\ \{A, B, G, K\} \vartriangle \{A, E, H, K\} \vartriangle \{C, G, K\} &= \{B, C, E, H, K\}, \text{ and} \\ \{A, E, H, K\} \vartriangle \{C, F, H, K\} \vartriangle \{C, G, K\} &= \{A, E, F, G, K\}. \end{aligned}$$

This corresponds to the fact that the first, third, and fifth rows from the matrix $\mathsf{C}(G)$ are linear combinations of other rows in $\mathbb{Z}_2$. Hence, we can remove them from $\mathsf{C}(G)$ without any loss of information. Remaining is a 5×9 matrix $\mathsf{C}_1(G)$ with its rows linearly independent over $\mathbb{Z}_2$. It is given by

$$\mathsf{C}_1(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Our next goal is to bring $\mathsf{C}_1(G)$ to the form in Equation (10.6). In general, this can be done by the Gauss-Jordan Method in linear algebra over $\mathbb{Z}_2$ transforming the matrix into reduced row-echelon form. Here we can first permute the columns to get $\mathsf{C}_2(G)$ given by

$$\mathsf{C}_2(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Adding the fourth row in $\mathsf{C}_2(G)$ to the first, we get $\mathsf{C}_3(G)$ given by

$$\mathsf{C}_3(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = [\mathsf{I}_5 \mid \mathsf{F}].$$

We note that there are no redundant cycles in matrix $\mathsf{C}_3(G)$, and $\mathsf{C}_3(G)$ is a fundamental cycle matrix of the required graph. Assume G has n vertices and m edges. Since the rank of $\mathsf{C}_3(G)$ is five and the network was assumed to be connected, we have the following information about the graph:

$$\begin{aligned} m &= 9, \\ m - n + 1 &= 5, \text{ and} \\ n - 1 &= 4. \end{aligned}$$

Constructing a graph from its incidence matrix is simple, but constructing a graph from its fundamental cycle matrix is difficult. We shall, therefore, construct an incidence matrix from $\mathsf{C}_3(G)$.

By Theorem 10.40 the rows of the incidence matrix $\mathsf{B}(G)$ are orthogonal to the rows in the cycle matrix $\mathsf{C}(G)$. By Definition 10.15 and Theorem 10.16 the rank of $\mathsf{B}(G)$ is here four. Hence, we must first look for a 9×4 matrix M, whose rows are linearly independent and are orthogonal to those of $\mathsf{C}_3(G)$.

Since $\mathsf{C}_3(G) = [\mathsf{I}_5 \mid \mathsf{F}]$, a matrix orthogonal to $\mathsf{C}_3(G)$ over $\mathbb{Z}_2$ is given by

$$\mathsf{M} = [-\mathsf{F}^t \mid \mathsf{I}_4] = [\mathsf{F}^t \mid \mathsf{I}_4].$$

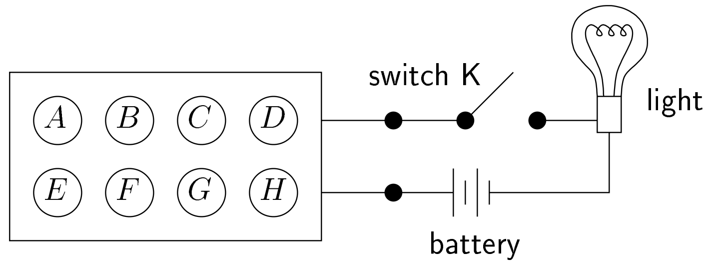

Figure 10.12: A black box with a switching network.

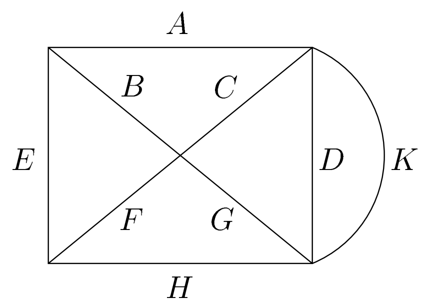

Figure 10.13: The graph G obtained from the incidence matrix $\mathsf{B}(G)$.

Thus

$$\mathsf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of M *is clearly four and one can check that indeed* $\mathsf{C}_3(G)\mathsf{M} = 0$*. Recall that adding one row to another row will not change the orthogonality of* M *to* $\mathsf{C}_3(G)$*.*

Before M *can be regarded as a reduced incidence matrix, it must have at most two ones in each column. This can be achieved by the operation of adding one row to another. Note that when we have added row* i *to another row then we cannot add row* i *to any other row since the identity matrix* I_μ *appears as a submatrix of* M*. We can in this case add the third row to the fourth in* M *and*

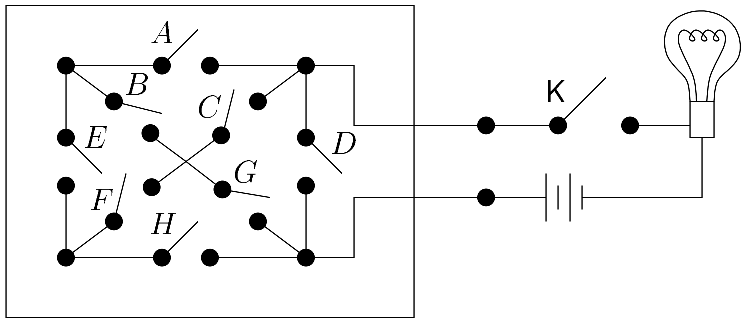

Figure 10.14: The switching network system inside the black box from Example 10.47 that corresponds to the graph G in Figure 10.13.

obtain

$$\mathsf{M}' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

in which no column contains more than two ones. This matrix M' *is the reduced incidence matrix. The actual incidence matrix* $\mathsf{B}(G)$ *of our graph* G *can be obtained by adding a fifth row to* M' *that is simply the sum of all the rows in* M'*. In this way there are exactly two ones in every column, which makes it an incidence matrix. We let*

$$\mathsf{B}(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

From $\mathsf{B}(G)$ *we have all of the information to construct our graph* G *and it is shown in Figure 10.13. The corresponding switching network is shown in Figure 10.14. From the raw information of what combination of switches turned on the lamp, we were able to in a systematic way to determine the switching network in the black box.*

10.8 Edge Cut Matrix

As we discussed in Chapter 6, cycles in a graph are closely related to edge cuts.

Definition 10.48
Let G be a graph with m edges and c edge cuts where the edges and the edge cuts are labeled as $E(G) = \{e_1, \ldots, e_m\}$ and $\mathcal{S}(G) = \{S_1, \ldots, S_c\}$, respectively. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, c\}$ define the s_{ij} by

$$s_{ij} = |S_i \cap \{e_j\}|.$$

That is, s_{ij} equals one if the edge cut S_i includes the edge e_j and is zero otherwise. The edge cut matrix *of G with respect to the labelings of $E(G)$ and* $\mathsf{S}(G)$ *is the $c \times m$ binary matrix*

$$\mathsf{S}(G) = [s_{ij}]_{i \in \{1,\ldots,c\}, j \in \{1,\ldots,m\}}.$$

The next example illustrates Definition 10.48.

☞ **Example 10.49**
Consider the graph G shown in Figure 10.15. It has eight edges and eight cutsets

$$\begin{aligned} E(G) &= \{e_1, \ldots, e_8\} \text{ and} \\ \mathcal{S}(G) &= \{S_1, \ldots, S_8\}. \end{aligned}$$

Here each S_i is given by the dotted lines in Figure 10.15. The edge cut matrix is given by

$$\mathsf{C}(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Remark 10.50

▷ *A column corresponding to a loop has only zeros in it.*

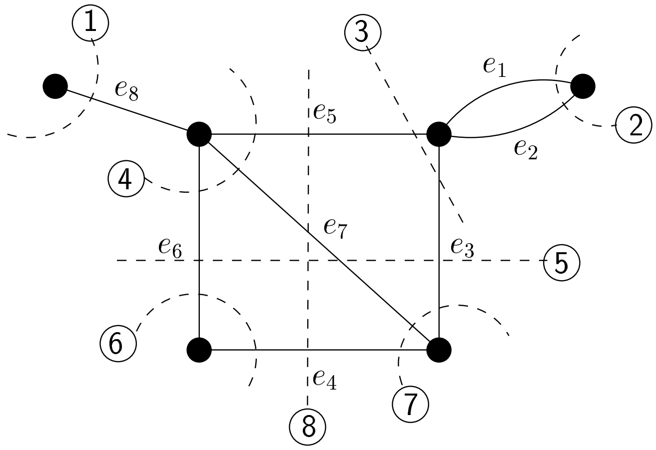

Figure 10.15: A graph G and its edge cut matrix.

- ▷ *Each row in* S(G) *is a vector corresponding to an edge cut.*
- ▷ *Parallel edges produce identical columns in the edge cut matrix .*
- ▷ *As with the incidence matrix and cycle matrix, the edge cut matrix* S(G) *depends on the labelings of the edges and the cycles. A permutation of rows or columns in an edge cut matrix simply corresponds to relabeling the cycles and edges.*
- ▷ *As for the cycle matrix, the edge cut matrix does not determine the graph. We discuss this further later in this section.*

By Theorem 6.6 every cycle has an even number of edges in common with each edge cut. Hence, every row in the cycle matrix C(G) is orthogonal to each row in the edge cut matrix S(G). Analogous to Theorem 10.40, we have the following observation.

Observation 10.51
Let G be a graph. For a given labeling of the edges, cycles, and edge cuts of G, computing in the Galois field $\mathbb{Z}_2$, we have that

$$\mathsf{C}(G)\mathsf{S}(G)^t = \mathsf{S}(G)\mathsf{C}(G)^t = 0.$$

By Corollary 9.55 we get the following result.

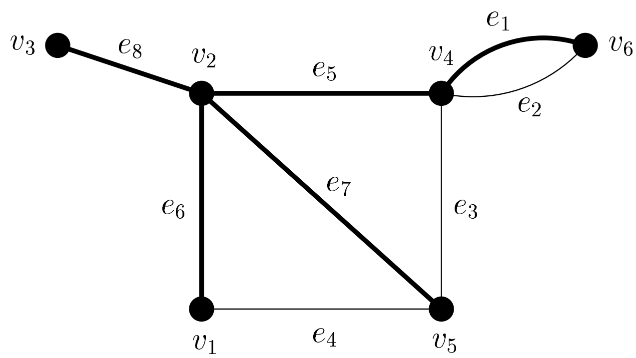

Figure 10.16: A spanning tree is shown by thick edges.

Theorem 10.52

For a graph G on n vertices, the edge cut matrix and the incidence matrix have the same rank:

$$\mathsf{rank}(\mathsf{S}(G)) = \mathsf{rank}(\mathsf{B}(G)) = n - 1.$$

By Theorem 10.52 we can remove all but $n-1$ rows from the edge cut matrix $\mathsf{S}(G)$ without losing information. Therefore, it is convenient to define a *fundamental edge cut matrix* $\mathsf{S}_f(G)$ as follows.

Definition 10.53

Let G be a graph on n vertices and m edges. Let T be a fixed spanning tree of G.

A fundamental edge cut matrix $\mathsf{S}_f(G)$ *with respect to the tree T is the the $(n-1) \times m$ submatrix of* $\mathsf{S}(G)$*, where the rows correspond to the set of fundamental edge cuts with respect to the tree T.*

Let I_{n-1} be the $(n-1) \times (n-1)$ identity matrix. By labeling the vertices in such a way that the first $m - n + 1$ vertices are the chords of the spanning tree T, the last $n-1$ vertices are the branches $\{b_1, \ldots, b_{n-1}\}$ of T, and the fundamental edge cuts with respect to T are $\{S_1, \ldots, S_{n-1}\}$, where each S_i is given by the branch b_i, we can write the fundamental edge cut matrix in the form

$$\mathsf{S}_f(G) = [\mathsf{S}_c(G) \mid \mathsf{I}_{n-1}]. \tag{10.8}$$

The next example illustrates Definition 10.53.

☞ **Example 10.54**

Consider the graph G shown in Figure 10.16 together with one of its spanning trees T that is shown by thick edges. Here the labeling in Figure 10.16 satisfies the conditions that yields the form of Equation (10.8). Hence, the fundamental edge cut matrix $\mathsf{S}_f(G)$ is given by

$$\mathsf{S}_f(G) = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = [\mathsf{S}_c(G) \mid \mathsf{I}_5].$$

Again note that in discussing edge cut matrices we have confined ourselves to connected graphs only. This treatment can be generalized to include disconnected graphs by considering one component at a time.

10.9 Relationships Among Fundamental Matrices

In this section we explore the relationships among the reduced incidence matrix $\mathsf{B}_f(G)$, the fundamental cycle matrix $\mathsf{C}_f(G)$, and the fundamental edge cut matrix $\mathsf{S}_f(G)$ of a connected graph with respect to a fixed spanning tree T of G. We assume that G has n vertices and m edges. Further, assume the labeling of the edges is such that the chords appear among the first $\mu = m - n + 1$ edges and the branches of T appear among the last $n - 1$ ones.

Partition $\mathsf{B}_f(G)$ into two submatrices as follows:

$$\mathsf{B}_f(G) = [\mathsf{B}_c(G) \mid \mathsf{B}_b(G)], \tag{10.9}$$

where $\mathsf{B}_b(G)$ consists of the $n - 1$ columns corresponding to the branches of T and $\mathsf{B}_c(G)$ is the remaining submatrix corresponding to the μ chords. Computing in $\mathbb{Z}_2$, we have by Theorem 10.40 the following:

$$0 = \mathsf{B}_f(G)\mathsf{C}_f(G)^t = [\mathsf{B}_c(G) \mid \mathsf{B}_b(G)] \begin{bmatrix} \mathsf{I}_\mu \\ \mathsf{C}_b(G)^t \end{bmatrix} = \mathsf{B}_c(G) + \mathsf{B}_b(G)\mathsf{C}_b(G)^t.$$

Thus we have $\mathsf{B}_b(G)\mathsf{C}_b(G)^t = \mathsf{B}_c(G)$. By Theorem 10.21 the matrix $\mathsf{B}_b(G)$ is nonsingular and hence its inverse exists. Therefore, we have

$$\mathsf{C}_b(G)^t = \mathsf{B}_b(G)^{-1}\mathsf{B}_c(G). \tag{10.10}$$

Similarly, by Observation 10.51 we have

$$0 = \mathsf{S}_f(G)\mathsf{C}_f(G)^t = [\mathsf{S}_c(G) \mid \mathsf{I}_{n-1}] \begin{bmatrix} \mathsf{I}_\mu \\ \mathsf{C}_b(G)^t \end{bmatrix} = \mathsf{S}_c(G) + \mathsf{C}_b(G)^t,$$

and so by the same token, we have

$$\mathsf{S}_c(G) = \mathsf{C}_b(G)^t = \mathsf{B}_b(G)^{-1}\mathsf{B}_c(G). \tag{10.11}$$

The next example makes these ideas concrete.

☞ **Example 10.55**

Consider the graph that we used in Figures 10.2, 10.15, and 10.16. Assume that the labelings of the vertices, edges, cycles, and edge cuts are as before in the corresponding Examples 10.12, 10.49, and 10.54 with respect to the given spanning tree T. Here we have that the reduced incidence matrix, the fundamental cycle matrix, and the fundamental edge cut matrix are given by the following:

$$\mathsf{B}_f(G) = \left[\begin{array}{ccc|ccccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] = [\mathsf{B}_c(G) \mid \mathsf{B}_b(G)],$$

$$\mathsf{C}_f(G) = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right] = [\mathsf{I}_3 \mid \mathsf{C}_b(G)], \text{ and}$$

$$\mathsf{S}_f(G) = \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = [\mathsf{S}_c(G) \mid \mathsf{I}_5].$$

Clearly, we have that

$$\mathsf{C}_b(G)^t = \mathsf{S}_c(G).$$

It is easy to verify that

$$\mathsf{C}_b(G)^t = \mathsf{B}_b(G)^{-1}\mathsf{B}_c(G).$$

These observations lead to three conclusions:

1. Given $\mathsf{B}_f(G) = [\mathsf{B}_c(G) \mid \mathsf{B}_b(G)]$ with respect to a spanning tree T of G, we can directly compute both $\mathsf{C}_f(G)$ and $\mathsf{S}_f(G)$.

2. Given either $\mathsf{C}_f(G)$ or $\mathsf{S}_f(G)$, we can compute the other. Thus, since $\mathsf{C}_f(G)$ determines a graph within 2-isomorphism, then so does $\mathsf{S}_f(G)$.

3. Given either $\mathsf{C}_f(G)$ or $\mathsf{S}_f(G)$, the incidence matrix $\mathsf{B}_f(G)$ cannot in general be completely determined. This is since a 2-isomorphism between graphs does not imply they are isomorphic.

10.10 Path Matrix

In the final section we discuss another binary matrix that in general does not store as much information about the graph as does the incidence matrix, the cycle matrix, and the edge cut matrix. Nonetheless, it is often used in certain communication and transportation networks.

Definition 10.56
Let G be a graph and $u, v \in V(G)$ be two vertices. Assume the vertices and the paths between u and v in G are labeled as $E(G) = \{e_1, \dots, e_m\}$ and $\mathcal{P}(u, v) = \{P_1, \dots, P_a\}$, respectively. For each $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, a\}$ define p_{ij} by

$$p_{ij} = |P_i \cap \{e_j\}|.$$

That is, p_{ij} equals one if the path P_i includes edge e_j and is zero otherwise. The path matrix *of G with respect to the above labelings of $E(G)$ and $\mathsf{P}(u, v)$ is the $a \times m$ binary matrix*

$$\mathsf{P}(u, v) = [p_{ij}]_{i \in \{1, \dots, a\}, j \in \{1, \dots, m\}}.$$

The next example illustrates Definition 10.56.

☞ **Example 10.57**

Consider the graph G shown in Figure 10.2. There are three paths between v_3 and v_4. Label them as

$$\begin{aligned} P_1 &= \{e_8, e_5\}, \\ P_2 &= \{e_8, e_7, e_3\}, \text{ and} \\ P_3 &= \{e_8, e_6, e_4, e_3\}. \end{aligned}$$

Here $\mathsf{P}(v_3, v_4)$ is a 3×8 binary matrix given by

$$\mathsf{P}(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

✎ **Remark 10.58**

▷ *A column containing only zeros corresponds to an edge that does not lie on path between u and v.*

▷ *A column containing only ones corresponds to a cut-edge that lies on every path between u and v.*

▷ *No rows contains only zeros.*

▷ *Note that the sum of two rows of $\mathsf{P}(u, v)$ corresponds to the symmetric difference of two paths between u and v in G. Such a set of edges either is a cycle or a disjoint union of cycles.*

The proof of the following theorem, describing a relation between the incidence matrix and the path matrix, is left as Exercise 40.

Theorem 10.59

If G is a connected graph with a labeling of its n vertices, m edges, and a paths between two given vertices $u, v \in V(G)$, then the $n \times a$ matrix $\mathsf{B}(G)\mathsf{P}(u, v)^t$ has ones in the two rows corresponding to the vertices u and v, and zeros everywhere else.

The next example illustrates Theorem 10.59.

☞ **Example 10.60**
Consider the incidence matrix for the graph G shown in Figure 10.2. Let $\mathsf{P}(v_3, v_4)$ be the path matrix with respect to vertices v_3 and v_4. In this case we have

$$\mathsf{B}(G)\mathsf{P}(v_3,v_4)^t = \begin{bmatrix} 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&1\\ 1&1&1&0&1&0&0&0\\ 0&0&1&1&0&0&1&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \cdot \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&1&1\\ 0&0&1\\ 1&0&0\\ 0&0&1\\ 0&1&0\\ 1&1&1 \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0\\ 0&0&0\\ 1&1&1\\ 1&1&1\\ 0&0&0\\ 0&0&0 \end{bmatrix}.$$

Other properties of the path matrix, such as the rank, are left for the reader to investigate.

We have seen there are several matrices that can be associated with graphs. Two of these, the incidence matrix and the adjacency matrix, describe a simple graph completely up to isomorphism. Two others, the cycle matrix and the edge cut matrix, display some important features of the graph and describe the graph only within 2-isomorphism. The path matrix contains the least information of these matrices. To look further into the structure of graphs, we investigated these matrices, pulled out submatrices, and examined the reduced incidence matrix, the fundamental cycle matrix, and the fundamental edge cut matrix with respect to some spanning tree of a graph. The properties brought out in this chapter do not, by any means, exhaust the list of possibilities. Many interesting and useful results are contained in the exercises of this chapter.

10.11 Exercises

1. For two given permutations σ and ρ on $\{1, \ldots, n\}$ show that the corresponding permutation matrices satisfy $\mathsf{P}_\sigma \mathsf{P}_\rho = \mathsf{P}_{\sigma\rho}$. In particular, show

that $\mathsf{P}_\sigma \mathsf{P}_{\sigma^{-1}} = \mathsf{I}_n$, the $n \times n$ identity matrix.

2. Use Exercise 1 to prove Observation 10.5.

3. Assume the graph G has two components G_1 and G_2. Show there is a labeling of the vertices of G such that the adjacency matrix of G has the form
$$\mathsf{A}(G) = \begin{bmatrix} \mathsf{A}(G_1) & 0 \\ 0 & \mathsf{A}(G_2) \end{bmatrix},$$
where 0 denotes rectangular matrices with zero in every entry. Generalize this result for G having an arbitrary number of components.

4. Let G be a simple graph. Show that the diagonal entry $a_{ii}^{(2)}$ in the square $\mathsf{A}(G)^2$ of the adjacency matrix of G with respect to the labeling
$$V(G) = \{u_1, \ldots u_n\}$$
is given by
$$a_{ii}^{(2)} = d_G(u_i).$$
Does this hold for general graphs? What can we say when every two distinct pairs of vertices have precisely k edges between them?

5. Show that each diagonal entry in X^3 equals twice the number of triangles passing through the corresponding vertex.

6. Complete the proof of Theorem 10.8.

7. Prove Corollary 10.9.

8. Prove Corollary 10.10.

9. Given the adjacency matrix $\mathsf{A}(G)$ of a graph G, how can you determine the diameter of the corresponding graph? [*Hint:* Consider a sum of the powers of $\mathsf{A}(G)$.]

10. Show that G has two components G_1 and G_2 if and only if there is a labeling of $V(G)$ and $E(G)$ such that the incidence matrix of G has the form
$$\mathsf{B}(G) = \begin{bmatrix} \mathsf{B}(G_1) & 0 \\ 0 & \mathsf{B}(G_2) \end{bmatrix}.$$
Generalize this result for G with k components and discuss the sizes of the blocks in $\mathsf{B}(G)$.

11. Suppose P and Q are $n \times n$ and $m \times m$ permutation matrices, respectively. Without using Observation 10.14 prove directly that $\mathsf{X}' = \mathsf{P}^t\mathsf{XQ}$ defines an equivalence relation among all $n \times m$ matrices.

12. Show that if P and Q are permutation matrices, then the rank of X and $\mathsf{P}^t\mathsf{XQ}$ is the same over any field.

13. Show that a disconnected graph on n vertices with k components has rank of $n - k$.

14. Use the Matrix-Tree Formula to compute $\tau(K_{2,n})$ for any n greater than zero. Can you find a combinatorial argument for your answer?

15. Use the Matrix-Tree Formula to compute $\tau(K_{3,n})$ for any n greater than zero.

16. Use the Matrix-Tree Formula to compute $\tau(K_{m,n})$ for any m and n greater than zero. The result should of course be symmetric in m and n.

17. Use the Matrix-Tree Formula to compute $\tau(K_n - e)$ for any n greater than one, where $K_n - e$ is a graph obtained from K_n by removing one edge.

18. Compute $\tau(W_n)$, where W_n is the *wheel* on $n+1$ vertices. The wheel W_n is C_n modified by adding one additional vertex that is adjacent to all of the vertices in C_n. When inserted in the middle of C_n, a wheel "forms." [*Hint:* If you use the Matrix-Tree Theorem, choose your cofactor carefully.]

19. Prove that the number of spanning trees in a connected graph equals the value of $\det(\mathsf{B}_f(G)\mathsf{B}_f)$, where this determinant is evaluated in the ring of integers $\mathbb{Z}$ and *not* in the Galois field $\mathbb{Z}_2$.

20. Compute the total resistance of the electrical network of resistors shown in Figure 10.17, where all the resistors are one ohm except the vertical resistor that is two ohm.

21. Compute the total resistance between u and v in the electrical network of resistors corresponding to the graph of the cube Q_3 shown in Figure 10.18, where each edge represent one ohm. (On numerous occasions this problem has been used as a brain teaser in college freshman physics classes. Without the machinery we have developed, one must rely on insight in electrical physics.)

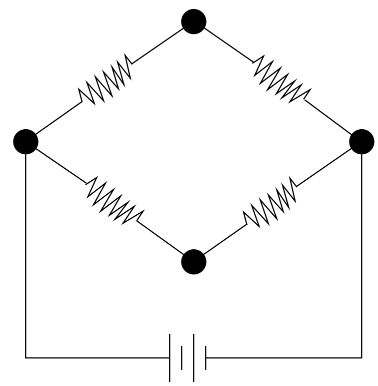

Figure 10.17: The four diagonal resistors are two ohms and the vertical resistor is one ohm.

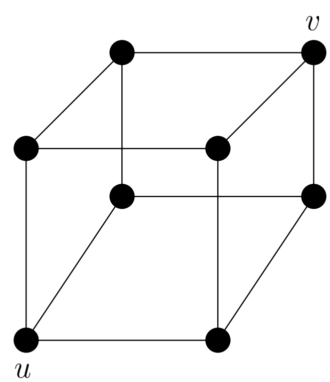

Figure 10.18: Each edge of the cube Q_3 represents one ohm.

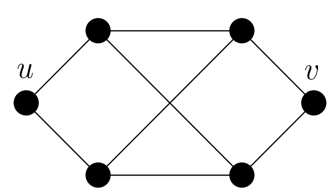

Figure 10.19: Horizontal resistors are one ohm; one diagonal resistor is two ohms; the other is three ohms.

22. Compute the total resistance in the electrical network of resistors shown in Figure 10.19, where all the resistors are one ohm except one diagonal resistor is two ohms while the other diagonal resistor is three ohms.

23. Let G be a graph with a labeling of its edges and cycles in such a way that $\mathsf{C}(G)$ has a block diagonal form as in Equation (10.5). Is G necessarily disconnected?

24. Recall that the *null space* of an $n \times n$ matrix A is the set of all vectors $\tilde{x}$ such that $\mathsf{A}(\tilde{x}) = \tilde{0}$, the zero vector. The *nullity* of A is the dimension of the null space and is denoted by null(A); it satisfies $\text{null}(\mathsf{A}) + \text{rank}(\mathsf{A}) = n$ (see [17]).

    Use this to prove the following theorem due to Sylvester.

    Theorem 10.61
    If A *is a* $k \times n$ *matrix and* B *is an* $n \times m$ *matrix then*

    $$\text{null}(\mathsf{AB}) \leq \text{null}(\mathsf{A}) + \text{null}(\mathsf{B}).$$

    [*Hint:* Form a basis for the null space of B and extend that basis to a basis for the null space of AB. Observe that the basis vectors you added are linearly independent and are mapped to the zero vector by A.]

25. Use Theorem 10.61 together with Theorem 10.40 to prove directly that for a graph G on n vertices and m edges we have $\text{rank}(\mathsf{C}(G)) \leq m - n + 1$.

26. Let n be a positive integer. What are the adjacency matrix and the incidence matrix for the null graph on n vertices? What about the complete graph on n vertices?

27. Consider the graph shown in Figure 10.20. With respect to the spanning tree $\{b, c, e, h, k\}$, write matrices $\mathsf{B}_f(G)$, $\mathsf{C}_f(G)$, and $\mathsf{S}_f(G)$ in the forms displayed in Example 10.55. Verify by actual computation Equations 10.10 and 10.11.

28. Show for a simple disconnected graph G with k components, n vertices, and m edges that the ranks of the matrices $\mathsf{B}_f(G)$, $\mathsf{C}_f(G)$, and $\mathsf{S}_f(G)$ are $n - k$, $m - n + k$, and $n - k$, respectively.

29. Label the edges of the graph G shown in Figure 6.10, then write down its cycle matrix $\mathsf{S}(G)$. Write the cycle matrix in a block form as in Equation (10.5).

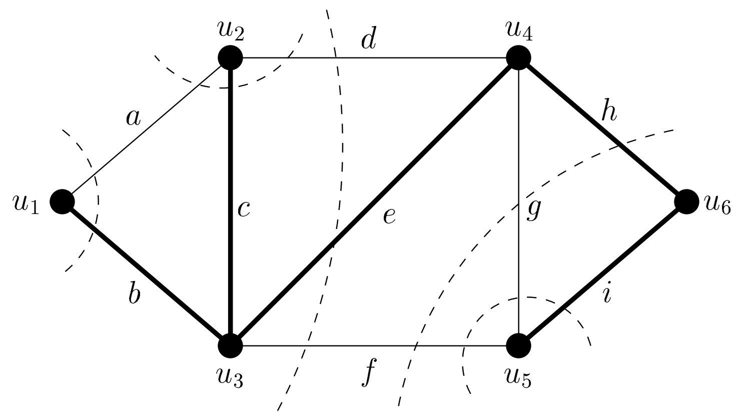

Figure 10.20: A graph with a spanning tree shown in bold.

30. Draw two nonisomorphic, connected, simple, and nonseparable graphs G_1 and G_2 with as small a number of edges as you can such that the cycle matrices $\mathsf{S}(G_1)$ and $\mathsf{S}(G_2)$ are equal. [*Hint:* G_1 and G_2 are 2-isomorphic, and must be 2-connected.]

31. A black box containing a switching network of seven switches labeled 1, 2, 3, 4, 5, 6, and 7 was subjected to the experiment shown in Example 10.47. The lamp was lit when each of the following combinations of switches was turned on, in addition to the external switch K: $\{1, 4, 5, \}$, $\{1, 4, 6, 7\}$, $\{2, 5, 7\}$, $\{2, 6\}$, $\{3, 5\}$, and $\{3, 6, 7\}$. Find the switching network configuration.

32. In Example 10.47 a graph was obtained corresponding to a given fundamental cycle matrix. Similarly, sketch a procedure for obtaining a graph if its fundamental edge cut matrix $\mathsf{S}_f(G)$ is given. Can you get two different (nonisomorphic) graphs for the same $\mathsf{S}_f(G)$? If yes, how are these different graphs related?

33. For a graph G let $\mathsf{S}(u, v)$ be the submatrix of $\mathsf{S}(G)$ containing only those rows of $\mathsf{S}(G)$ that represent edge cuts with respect to vertices u and v. Show that $\mathsf{S}(u, v)$ contains a fundamental edge cut matrix $\mathsf{S}_f(G)$ of G.

34. Show that you can determine a graph within 2-isomorphism if you were given the set of all its spanning trees. [*Hint:* Theorem 6.5 relates spanning

trees and edge cuts. The set of all edge cuts determines a graph within 2-isomorphism.]

35. If the following is the list of all spanning trees of a graph G, determine G.

$$\{a,c,d,e\},\quad \{a,c,d,f\},\quad \{b,c,d,e\},\quad \{b,c,d,f\},$$
$$\{a,c,e,f\},\quad \{b,c,e,f\},\quad \{a,d,e,f\},\quad \{b,d,e,f\},$$
$$\{a,b,d,e\},\quad \{a,b,d,f\},\ \text{and}\quad \{a,b,e,f\}.$$

36. Express the relationship of dualism between two planar, simple graphs in terms of appropriate matrices.

37. Characterize simple, self-dual graphs in terms of their cycle and edge cut matrices.

38. Prove the following equalities:

$$\begin{aligned} \mathsf{C}_f(G) &= [\mathsf{I}_\mu \mid \mathsf{B}_c(G)^t(\mathsf{B}_b(G)^{-1})^t] \text{ and} \\ \mathsf{S}_f(G) &= \mathsf{B}_b(G)^{-1}\mathsf{B}_f(G). \end{aligned}$$

39. Write down the path matrix $\mathsf{P}(v_1, v_6)$ for the graph shown in Figure 6.3. Verify the remarks given right after Example 10.57 and Theorem 10.59.

40. Prove Theorem 10.59.

41. Find the adjacency matrix, reduced incidence matrix, fundamental cycle matrix, and edge cut matrix for the complete bipartite graph $K_{m,n}$ with respect to a suitable spanning tree.

42. Label the edges of the graph G shown in Figure 6.10 and write its adjacency matrix $\mathsf{A}(G)$. How does the fact that the graph is separable reflect in $\mathsf{A}(G)$? Characterize the adjacency matrix of a separable graph G in general.

43. Similar to the cycle or edge cut matrix, define a *spanning-tree matrix* for a connected graph and one of its spanning trees. Observe some of the properties of this matrix.

44. Let u and v be a pair of vertices in a simple nonseparable graph G, and let $\mathsf{P}(u,v)$ be the corresponding path matrix of G. Prove that every cycle in G is obtained as a sum of two rows of $\mathsf{P}(u,v)$ over the Galois field $\mathbb{Z}_2$.

From this result, prove that a path matrix in a simple nonseparable graph determines the graph within 2-isomorphism. [*Hint:* Every cycle γ in G falls in one of three categories: (1) γ passes through both u and v; (2) γ passes through neither u nor v; or (3) γ passes through either u or v. Consider all three cases, and use Theorem 6.36.]

45. Prove that for a connected graph G without loops, the subspaces $W(\mathcal{C}_G)$ and $W(\mathcal{S}_G)$ are orthogonal complements of $W(G)$ over the Galois field $\mathbb{Z}_2$ if and only if the number of spanning trees in G is odd. [*Hint:* If G has m edges, define a new $m \times m$ matrix

$$\mathsf{M}(G) = \begin{bmatrix} \mathsf{S}_f(G) \\ \mathsf{C}_f(G) \end{bmatrix}.$$

Compute $\det(\mathsf{M}(G)\mathsf{M}(G)^t)$ by using the identity

$$\mathsf{S}_f(G)\mathsf{C}_f(G)^t = \mathsf{C}_f(G)\mathsf{S}_f(G)^t = 0$$

and the Binet-Cauchy Theorem. Show that

$$\det(\mathsf{M}(G)\mathsf{M}(G)^t) = 1$$

in the Galois field $\mathbb{Z}_2$ if and only if G has an odd number of spanning trees.]