Planar Graphs

Chapter Goals

- Define planar graphs, and discuss graphs embeddable in various surfaces.
- Present Euler's Formula for planar graphs and its consequences.
- Present a characterization of planar graphs in the Kuratowski-Wagner Theorem.
- Define homeomorphic graphs, contraction, and minors of graphs.

7.1 Introduction

Throughout this book we have represented vertices by dots and edges by lines or curves. The dots help us distinguish vertices from the intersection of edges. The graph in Figure 1.17 has four vertices. Its edges e and f intersect in the figure at a point in the plane that does not represent a vertex. Some graphs may be drawn in the plane so that any intersection of edges is a vertex. Other times, we may try to draw a given graph in this manner without any success. Despite our best efforts, it may be difficult to determine if this is possible. In this chapter we address the following important question: "Is it possible to draw a graph G in a plane without *edge crossings*?" By edge crossings we mean intersections of edges that are not vertices.

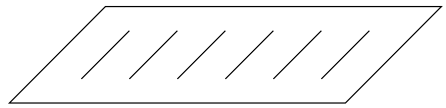

Figure 7.1: The plane surface P.

In §7.2 we provide basic definitions and a discussion about embeddings in surfaces. We prove a graph is planar if and only if it can be embedded in a sphere. In §7.3 we examine additional properties of planar embeddings. §7.4 describes Euler's formula for planar graphs and its various consequences. Homeomorphic graphs, contraction, and minors are examined in §7.5. In §7.6 we present part of the proof of the Kuratowski-Wagner Theorem that characterizes planar graphs.

7.2 Embeddings in Surfaces

The planarity question is of great significance in graph theory from a theoretical and a historical point of view. In addition, planarity and other related concepts are useful in many practical situations. For instance in the design of circuits, electrical engineers want to minimize the number of layers. Another example is the "Three Utilities Puzzle," Problem 1.3. Solving this puzzle is equivalent to determining if a particular graph can be drawn in a plane without edge crossings. Before we attempt to "draw a graph in the plane," though let us examine the meaning of "drawing" a graph in the plane and various surfaces more closely. In the following, we assume familiarity with the set of real numbers $\mathbb{R}$, and the two and three dimensional Cartesian products, $\mathbb{R}^2$ and $\mathbb{R}^3$.

The next convention addresses the four types of surfaces we talk about in this book.

✎ **Convention 7.1**
Let g be a natural number. We have the plane surface $P = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, the sphere surface $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$*, the* torus *surface T denoting the usual torus and embedded in $\mathbb{R}^3$, and the g-*torus *surface T_g embedded in $\mathbb{R}^3$.*

Remark 7.2
The usual torus T is really T_1 in the general setting. The surface T_g is a g-fold torus with g holes in it.

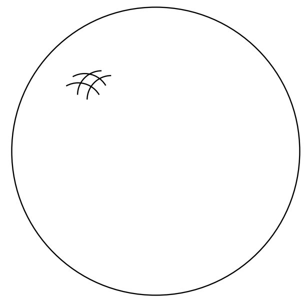

Figure 7.2: The sphere surface S^2.

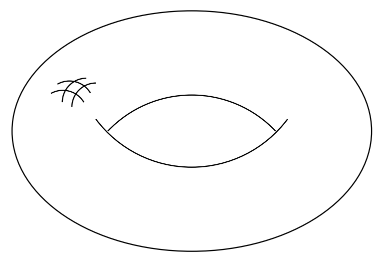

Figure 7.3: The torus T.

Figures 7.1, 7.2, 7.3, and 7.4 depict these four surfaces. The next definition makes precise the notion of drawing a graph in a particular type of surface.

Definition 7.3
Let G be a graph and S a surface.

- ▷ *We say that G is* embeddable *in S if G has a representation in S in the following way:*
  1. *The vertices in $V(G)$ are represented by distinct points in S.*
  2. *The edges in $E(G)$ are represented by distinct continuous curves $[0; 1] \to S$.*
  3. *Curves representing the edges can only intersect each other at their endpoints.*
- ▷ *We say that G is* planar *if it can be embedded in the plane P.*
- ▷ *A graph G is planar if it can be drawn in the plane with no edge crossings.*

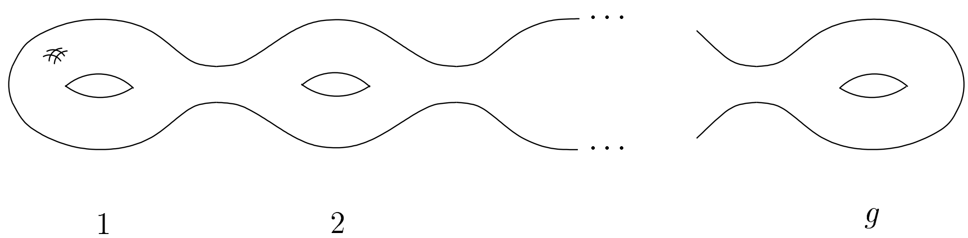

Figure 7.4: The g-torus T_g.

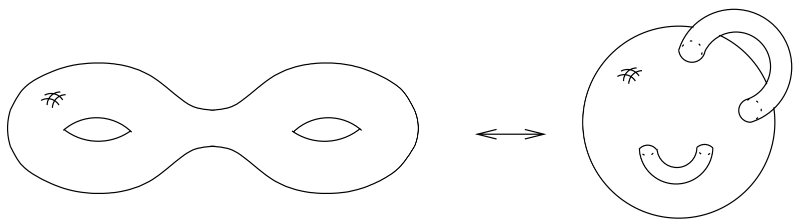

Figure 7.5: The 2-torus can be viewed as a sphere with two handles attached to it.

▷ *A* plane graph *of G is a drawing of G such that there are no edge crossings.*

Remark 7.4

1. *Since our surface always sits in the real three dimensional space $\mathbb{R}^3$, a curve $\gamma : [0;1] \to S$ is continuous if and only if it is continuous when we view it as a function mapping $[0;1]$ to $\mathbb{R}^3$.*

2. *If $\gamma : [0;1] \to S$ is a curve that represents an edge, then the endpoints of this curve, $\gamma(0)$ and $\gamma(1)$, represent the endvertices of the corresponding edge.*

3. *One can view the 2-torus as a sphere where we have added two "handles" on it as shown in Figure 7.5. One can deform the g-torus into a sphere with g handles in a continuous fashion (that is, without cutting the surface nor closing any holes) and vice versa. Hence, a graph is embeddable in T_g if and only if it is embeddable in a sphere with g attached handles.*

For the most part, we will be concerned with planar graphs and their embeddings in the plane P. For planar graphs we have the following theorem.

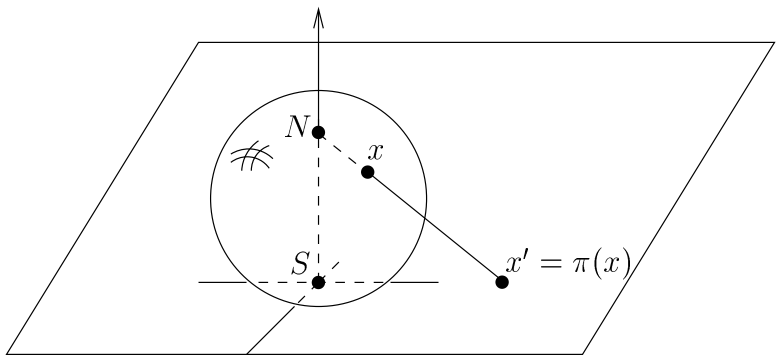

Figure 7.6: The stereographic projection.

Theorem 7.5
A graph is planar if and only if it can be embedded in the sphere S^2.

PROOF: Imagine that we have the sphere S^2 lying on the plane P as shown in Figure 7.6. The sphere and the plane have exactly one point in common. This point is called the "south pole," and is denoted by S. The opposite point of the sphere is called the "north pole," and is denoted by N.

For each point x on the sphere, not including the north pole, the line segment from N through x intersects the plane P in exactly one point x'. Consider the *stereographic projection* π defined by

$$\begin{aligned} \pi : S^2 \setminus \{N\} &\rightarrow P, \text{ where} \\ \pi(x) &= x'. \end{aligned}$$

This map is bijective, and moreover, both π and its inverse π^{-1} are continuous. This implies that if ξ is a continuous curve in S^2, then $\pi(\xi)$ is a continuous curve in P and vice versa.

If G is a graph that has an embedding G' in S^2, then clearly we can move the embedding G' in such a way that G' is contained in $S^2 \setminus \{N\}$. In this case $\pi(G')$ is an embedding of G in the plane P. Thus G is planar. Likewise, if G has a planar embedding G'' in the plane P, then $\pi^{-1}(G'')$ is an embedding of G in $S^2 \setminus \{N\}$. In particular, this is an embedding of G in S^2. Hence, we have the theorem. $\square$

Theorem 7.5 shows that for our purposes the plane P and the sphere S^2 are equivalent.

Assume that a graph G is embeddable in a surface S. In this case we have an embedding G' of G in S. All of the graph-theoretic information of G can be interpreted from the geometric embedding G'. Strictly speaking, G and G' are not the same thing since G is an abstract graph and G' is a subset in $\mathbb{R}^3$.

✎ **Convention 7.6**
For a graph G that is embeddable in a surface S, we call the actual embedding G' of G in S the surface-S graph *of G, or just a* surface graph *when the type is not important. In particular, if S is the plane P then we call the embedding a* plane graph. *In this case we talk about plane G instead of a graph G together with a plane embedding G'.*

7.3 More on Planar Embeddings

Let S be a surface. As one can imagine, a continuous curve $\gamma : [0;1] \to S$ can be arbitrarily long. It can have many twists and turns, so the image of a continuous curve in the plane P can be quite complex. Hence, it may have appeared that the ability to draw a planar graph in a plane depended on the ability to draw many crooked lines through devious routes. This is not the case. The following important and somewhat surprising result of Fary [11] tells us that there is no need to bend edges to avoid edge crossings in drawing a planar graph.

Theorem 7.7 (Fary Embedding)
Any simple planar graph G can be embedded in the plane such that every edge is drawn as a straight line segment.

We omit the proofs of the theorems in this section as they often require technical facts from real analysis that we do not need elsewhere in this book.

Definition 7.8
An embedding of a simple planar graph in the plane such that every edge is drawn as a straight line is called a Fary embedding.

➤ **Note 7.9**
Recall that if $\tilde{a} = (a_1, a_2)$ and $\tilde{b} = (b_1, b_2)$ are two points in the plane, then the line segment between $\tilde{a}$ and $\tilde{b}$ can be given by

$$\gamma(t) = (1-t)\tilde{a} + t\tilde{b} = ((1-t)a_1 + tb_1, (1-t)a_2 + tb_2).$$

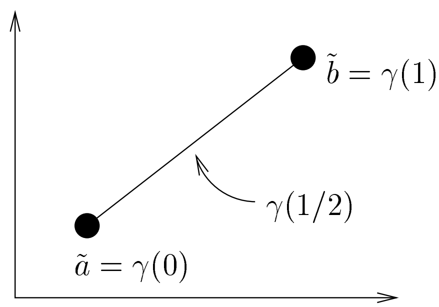

Figure 7.7: The line segment from $\tilde{a}$ to $\tilde{b}$ in the plane.

Here, $\gamma(0) = \tilde{a}$ *and* $\gamma(1) = \tilde{b}$ *as illustrated in Figure 7.7.*

The next example depicts a Fary embedding of a graph.

☞ **Example 7.10**
Consider the two plane graphs in Figure 7.8. They both are embeddings of the same graph, but the one on the right has all of its edges represented by straight line segments. This is a Fary embedding.

From a given plane graph G we can get all of the graph-theoretic information we need. However, we actually get much more since G is a subset of the plane P, which has many properties of its own. The following definition leads us in this direction.

Definition 7.11
Let f *be a natural number greater than zero. If* G *is a plane graph in* P*, then there are regions* $F_1, \ldots, F_f$ *in* P *such that* $P \setminus G = F_1 \cup \cdots \cup F_f$ *and the following conditions are satisfied:*

1. *The sets* $F_1, \ldots, F_f$ *are pairwise disjoint. That is,* $F_i \cap F_j$ *is empty for all* i *not equal to* j*.*
2. *For each* $i \in \{1, \ldots, f\}$ *and every two points* $x, y \in F_i$*, there is a continuous curve from* x *to* y *that does not intersect* G*.*
3. *Each continuous curve from* $x \in F_i$ *to* $y \in F_j$*, where* i *does not equal* j*, intersects* G *at some point.*

Each region F_i *is called a* face *of the plane graph* G*.*

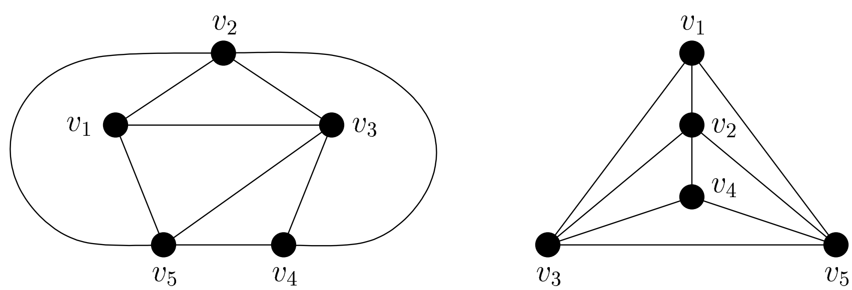

Figure 7.8: The plane graph on the right is a Fary embedding of the graph on the left.

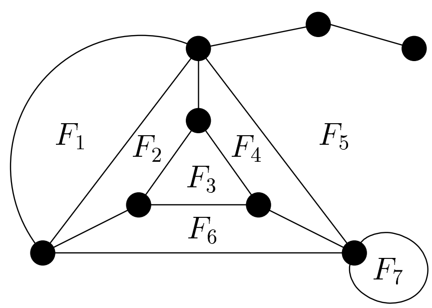

Figure 7.9: The plane graph G has seven faces: $F_1, \ldots, F_7$.

➤ **Note 7.12**

Definition 7.11 can made for any surface S in $\mathbb{R}^3$. That is, we can talk about faces of sphere graphs, torus graphs, and general g-torus graphs in the same way as we do for plane graphs.

The next example illustrates Definition 7.11.

☞ **Example 7.13**

Consider the plane graph shown in Figure 7.9. It has 8 vertices, 13 edges, and 7 faces. Note that $8-13+7=2$ for this plane graph. We will see in Theorem 7.19 that this relation between the number of vertices, edges, and faces of a plane graphs always holds. Any embedding of the underlying graph of a plane graph always has the same number of faces. In our case there are seven.

➤ **Note 7.14**

In the plane graph in Example 7.13 there is exactly one face, namely F_5, that is unbounded*. By this we mean that it can contain the interior region of a* *rectangular square of an arbitrary size. This face is called the* infinite face *of the plane graph. The other faces are all bounded and can each be contained within a square of a certain size.*

Consider a plane graph G in the plane P with f faces $F_1, \ldots, F_f$, where F_1 is the infinite face. Let $i \in \{1, \ldots, f\}$ and let us put the sphere S^2 on P such that the south pole of S^2 touches a point inside the face F_i. Now apply the inverse stereographic projection and obtain an embedding of G on the sphere S^2. Next roll the sphere S^2, with the embedding of G on it, in such a way that the former south pole becomes the north pole. At this point apply the stereographic projection and get a plane graph G', which is isomorphic to our original plane graph G. In this new plane graph G' the face F_i has become the infinite face and the earlier infinite face F_1 has become a bounded face. From this we obtain the following.

Theorem 7.15
Every planar graph G can be embedded in the plane in such a way that any specified face can be made the infinite face.

➤ **Note 7.16**
For all surfaces S, other than the plane, the faces of a surface-S graph are all bounded in the sense that they can each be contained in a box of a certain size. In particular, every face of a graph embedded in a torus is bounded.

The next theorem will be useful later in the chapter.

Theorem 7.17
For a plane graph G we have the following:

1. *A tree has only one face.*
2. Jordan Curve Theorem *(restricted version): If G is a plane embedding of a cycle graph, then G has precisely two faces. One face is formed by the region "inside" the cycle and one face is formed by the region "outside" the cycle.*

Next is the general version of the Jordan Curve Theorem, which is hard to prove in a precise manner.

Theorem 7.18 (Jordan Curve)
If $\gamma : [0;1] \to P$ is a continuous and closed *curve (That is, $\gamma(0) = \gamma(1)$ such that γ restricted to $[0;1[= [0;1] \setminus \{1\}$ is an injective map.) then $P \setminus \gamma([0;1])$ is a disjoint union of two connected regions S_1 and S_2 (see [24] for further details).*

7.4 Euler's Formula and Consequences

One of the most important properties of planar graphs is given in Theorem 7.19. This result is due to Leonhard Euler.

Theorem 7.19 (Euler's Formula)
For a connected plane graph with n vertices, e edges, and f faces we have

$$n - e + f = 2.$$

PROOF: We use induction on f, the number of faces of G, to prove the result.

If f equals one then G cannot contain any cycles. Since G is connected, it must be a tree by definition. By Theorem 4.7 e equals $n-1$ and we have

$$n - e + f = n - (n-1) + 1 = 2.$$

If f is greater than one then by Theorem 7.17(1) G cannot be a tree, and hence has a cycle. Let c be an edge on the cycle. By Theorem 7.17(2) c is on the boundary of two distinct faces S and S'. By removing the edge c, the two faces S and S' merge and form a new face S''. Since $G \setminus c$ now has n vertices, $e' = e-1$ edges, and $f' = f-1$ faces, the induction hypothesis applies to $G \setminus c$. We have

$$2 = n - e' + f' = n - e + f$$

completing the proof of the theorem. □

Euler's Formula generates some interesting corollaries.

Corollary 7.20
Let G be a simple planar graph on n vertices having e edges. The following properties hold

1. *The number of edges e is less than or equal to $3n - 6$.*

2. If G has no 3-cycles, then e is less than or equal to $2n-4$.

PROOF: Let $F_1, \ldots, F_f$ be the faces of a plane embedding of G. If f equals one then G is a forest and hence e is less than or equal to $n-1$. This implies the corollary. Otherwise, let e_i be the number of edges that bound the face F_i for each $i \in \{1, \ldots, f\}$. Since each edge of G bounds either one or two faces, when we sum all of the e_i edges, we count each edge in G either once or twice. Hence, we have

$$\sum_{i=1}^{f} e_i \leq 2e. \tag{7.1}$$

Each bounded face F_i is bounded by at least three edges since G is simple. Therefore, we have that e_i is greater than two for each i. So, we get

$$\sum_{i=1}^{f} e_i \geq 3f. \tag{7.2}$$

From Inequalities (7.1) and (7.2), we have

$$3f \leq 2e. \tag{7.3}$$

By Inequality (7.3) and using Theorem 7.19, we obtain

$$e = 3e - 2e \leq 3(e-f) = 3(n-2) = 3n-6,$$

proving the first inequality of the theorem.

If G has no 3-cycles, then each face F_i is bounded by at least four edges. Therefore, we have e_i is greater than three for each i. Hence, Inequality (7.2) becomes

$$\sum_{i=1}^{f} e_i \geq 4f.$$

From this and Inequality (7.1), we get $4f \leq 2e$ so $2f \leq e$. This coupled with Theorem 7.19 yields

$$e = 2e - e \leq 2(e-f) = 2(n-2) = 2n-4,$$

proving the second inequality and thereby completing the proof. □

The next example illustrates how Corollary 7.20 can be used to prove that certain graphs are not planar.

Example 7.21

Consider the complete graph K_5. Since it has n equals five vertices and $e = 5(5-1)/2 = 10$ edges, we have that

$$e = 10 > 9 = 3n - 6.$$

By Corollary 7.20, K_5 is not be planar.

In the same manner consider the complete bipartite graph $K_{3,3}$. Since it has n equals six vertices and $e = 3 \cdot 3 = 9$ edges, we have

$$e = 9 > 8 = 2n - 4.$$

Since $K_{3,3}$ has no 3-cycles, it is not planar either.

This solves Problem 1.3.

As we shall see later, the graphs K_5 and $K_{3,3}$ are in some sense the fundamental nonplanar graphs. For emphasis we make the following observation as proved in Example 7.21.

Observation 7.22

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are both nonplanar.

Remark 7.23

Clearly, every subgraph of a planar graph is also planar. By Observation 7.22 we have that for n greater than four the complete graph K_n is also nonplanar. This is because it contains the nonplanar graph K_5 as a subgraph. Also, for m and n greater than two, the complete bipartite graph $K_{m,n}$ is also nonplanar. This is since it contains $K_{3,3}$ as a subgraph. However, the graphs K_4 and $K_{2,n}$ are *planar as shown in Figure 7.10.*

If a planar graph G on n vertices has exactly $3n-6$ edges, then equality must hold in Inequalities (7.1) and (7.2). This means that each face of G is bounded by a *triangle* (see Exercise 7). A triangle is another word for 3-cycle. Clearly, the other direction holds as well. In this case we say that G is a *maximal planar graph* or a *triangulation*.

The following corollary illustrates another interesting property of simple, connected planar graphs.

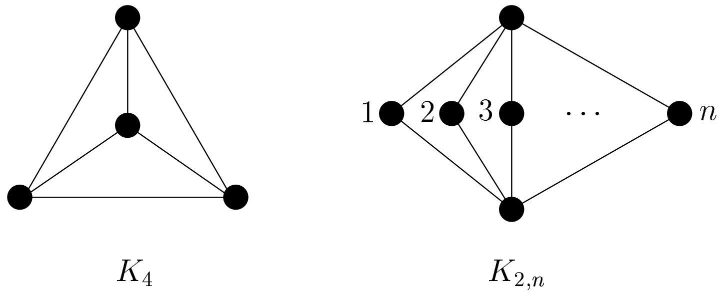

Figure 7.10: K_4 is planar. $K_{2,n}$ is planar for all n greater than zero.

Corollary 7.24
Every simple, connected planar graph contains a vertex of degree five or less.

PROOF: Let G be a simple, connected planar graph on n vertices and with e edges. Let δ be the least degree of all of the vertices of G. Then, by the Hand Shaking Theorem 1.24 and Corollary 7.20, we have

$$n\delta \le \sum_{u \in V(G)} d_G(u) = 2e \le 2(3n-6) < 6n.$$

Hence, δ is a natural number less than six. This proves the corollary. $\square$

We will use Corollary 7.24 in many inductive proofs regarding vertex *colorings* of planar graphs.

Here is an important question: Is there an automatic method or procedure one can apply to determine whether a graph is planar or not? Clearly if a graph on n vertices has more than $3n-6$ edges, then by Corollary 7.20 it cannot be planar. But needless to say, there are many nonplanar graphs on n vertices with $3n-6$ or fewer edges. We address this important question further in the sections to come. We conclude this section with a theorem showing that when determining the planarity of a graph, it suffices to consider graphs with no cut-edges and no cut-vertices. That is, we can assume the graph is 2-connected.

Theorem 7.25
A connected graph G is planar if and only if each block of G is planar.

PROOF: We prove this by induction on the number of blocks of G. If G is 2-connected, then G has only one block consisting of G itself. So, the theorem is a tautology.

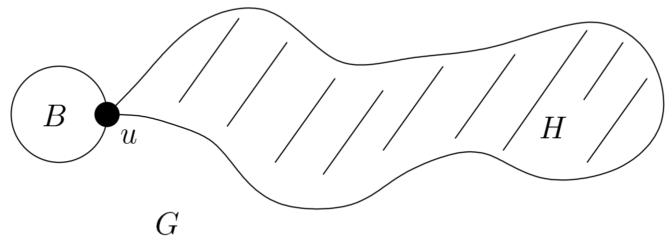

Figure 7.11: The graph G is planar if and only if both the block B and the subgraph H are planar.

Assume that G has more than one block. Let B be a block that corresponds to a leaf in the block-cutpoint graph $BC(G)$ of G. Let H be the union of the other blocks. We now have

$$\begin{aligned} B \cup H &= G \text{ and} \\ B \cap H &= \{u\}, \end{aligned}$$

where u is a cutpoint that is in B. Now, G is planar if and only if both B and H are planar (see Figure 7.11).

Since H has one fewer blocks than G, the induction hypothesis applies. We conclude that H is planar if and only if all the blocks of H are planar. Because the set of blocks of G consists of B together with the set of blocks of H, we have the theorem. $\square$

7.5 Characterization of Planar Graphs

In the following sections we describe a way to detect exactly when a given graph is planar. These results, due to Kuratowski and Wagner, are considered one of the corner stones of graph theory. Before presenting them, some new definitions are needed. Although intuitively clear, these ideas are somewhat lengthy when defined in mathematical language.

Initially, there are a couple of reductions one can do when we are given a graph, bearing in mind Theorem 7.25 from the previous section. We number the reductions as follows:

1. A graph is planar if and only if each of its components is planar. Hence, it suffices to consider connected graphs.
2. A connected graph is planar if and only if each of its block is planar. Hence, it suffices to consider 2-connected graphs.
3. A 2-connected graph G is planar if and only if the derived graph G' is planar, where G' is the simple graph formed from G by removing all the loops from G and by replacing all multiple edges in G by a single edge.
4. The graph G is planar if and only if the derived graph G'' is planar. The graph G'' is formed from the simple graph G', defined in the previous step, by eliminating all vertices of degree two and merging their incident edges. That is, replace all paths (u, w, v) of length two, where $d_{G'}(w)$ equals two, by a single edge $\{u, v\}$. If G'' is not a simple graph, repeat steps 3 and 4 until it becomes simple.

Usually, the application of the procedure just described reduces our graph quite substantially. This is because we can assume the graph is simple, has at least five vertices, is 2-connected, and contains only vertices of degree three or more. But even with these assumptions, it is not obvious when a graph is planar.

Look at step number four of our procedure. This reduction in reverse gives rise to the following.

Definition 7.26
Let G be a graph and $e = \{u, v\}$ an edge of G. Let G' be the graph formed from G by replacing the edge e by a simple path $(u_0, u_1, \ldots, u_k)$, where u_0 equals u and u_k equals v. We say that G' is a subdivision *of G.*

Two graphs G' and G'' are homeomorphic *or* topologically equivalent *if both G' and G'' are subdivisions of the same graph G.*

Figure 7.12 illustrates the concept of subdivision.

➤ **Note 7.27**
If G is a planar graph, then so is any subdivision G'. We can obtain the graph G back from G' by a sequence of simple contractions, where we contract every subdivided edge back to a single edge. Since any simple contraction of a planar graph is planar, we see that if G' is planar so is G. We summarize in the following observation.

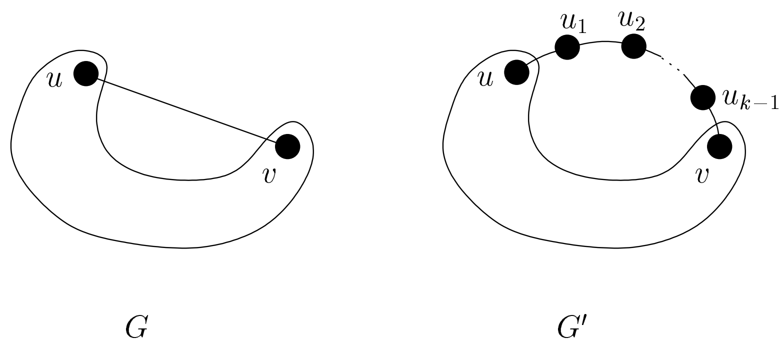

Figure 7.12: The graph G' is a subdivision of the graph G.

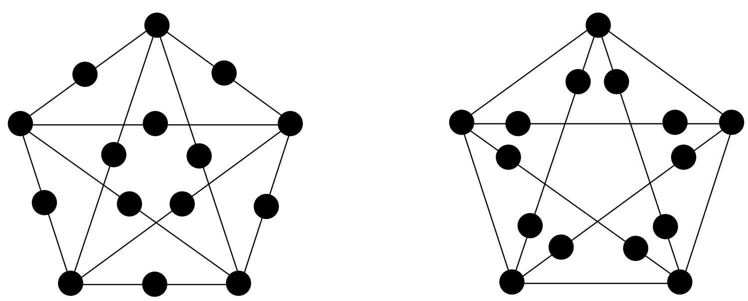

Figure 7.13: Homeomorphic graphs.

Observation 7.28
A graph G is planar if and only if every graph homeomorphic to G is planar.

The next example illustrates these ideas.

☞ **Example 7.29**
The two graphs shown in Figure 7.13 are homeomorphic since they are both subdivisions of K_5.

➤ **Note 7.30**
If G and G' are two homeomorphic graphs, then there is a graph G'' that is both a subdivision of G and also of G'. This property is equivalent to the definition of homeomorphism (see Exercise 14).

The next example illustrates non-homeomorphic graphs.

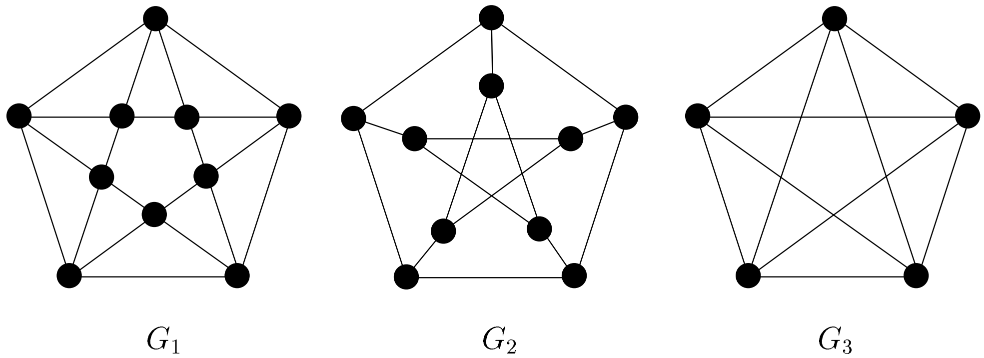

Figure 7.14: Three non-homeomorphic graphs.

Example 7.31

In contrast to the graphs shown in Figure 7.13 none of the three graphs G_1, G_2, nor G_3 shown in Figure 7.14 are homeomorphic.

Subdivision does not change the number of vertices with degree not equal to two. Hence, if two graphs are homeomorphic, they must have the same number of vertices of degree i for each $i \in \mathbb{N} \setminus \{2\}$. Thus none of the graphs G_1, G_2, and G_3 are homeomorphic: The graph G_1 has all of its ten vertices of degree four, the Petersen graph G_2 has no vertices of degree four, and the complete graph G_3 has five vertices of degree four.

As we saw earlier in Observation 7.22, neither K_5 nor $K_{3,3}$ are planar graphs. Also, any graph containing K_5 or $K_{3,3}$ as subgraphs cannot be planar since every subgraph of a planar graph is also planar. By Observation 7.28, any graph homeomorphic to either K_5 or $K_{3,3}$ is not planar. Therefore, any graph containing a subgraph that is homeomorphic to either K_5 or $K_{3,3}$ is not planar either. Remarkably, this is precisely the *characterization* of planar graphs as the following theorem by Kuratowski from 1930 states. Kuratowski's proof [22] is fairly elementary but long. It is based on an exhaustive case analysis. Many different proofs have appeared since and a readable proof can be found in [31].

Theorem 7.32 (Kuratowski's)

A graph G is planar if and only if it has no subgraphs homeomorphic to either K_5 or $K_{3,3}$.

We have that the operation of subdivision preserves planarity of graphs. Also, a simple contraction of an edge in a planar graph yields a planar graph.

Definition 7.33
For a simple and connected graph G, we say that the simple graph G' is a contraction *of G if G' can be obtained from G by a sequence of simple contractions. That is, if there is a finite sequence $G_0, G_1, \ldots, G_k$ such that*

1. *$G = G_0$ and $G' = G_k$,*
2. *$G_{i+1} = G_i/e_i$ for some edge e_i in G_i, where $i \in \{0, 1, \ldots, k-1\}$.*

If G' is a contraction of a subgraph of G, then we say that G' is a minor *of G.*

The following example provides a useful way of viewing minors.

☞ **Example 7.34**
Given a simple and connected graph G, we can think of any contraction G' of G as follows. The vertices $V(G') = \{u'_1, \ldots, u'_m\}$ correspond to a partition of $V(G)$ into disjoint subsets $X_1, \ldots, X_m$ such that

1. *Each X_i induces a connected subgraph $G[X_i]$ of G, and*
2. *The vertices u'_i and u'_j are connected by an edge in G' if and only if there is some edge in G with one endvertex in X_i and the other in X_j.*

On the left hand side of Figure 7.15 is a graph G together with a partition of its vertices. The righthand side of the figure shows a minor of G. Notice the correspondence between the $G[X_i]$'s and the u'_i's.

At this point we have two operations, subdivision and contraction, that preserve planarity. The following observation, whose proof is an easy exercise, summarizes.

Observation 7.35
Let G be a simple connected planar graph.

1. *Each graph G' homeomorphic to a subgraph of G is planar.*
2. *Each graph G'' that is a minor of G is planar.*

We also have the following observation, whose proof is required in Exercise 15.

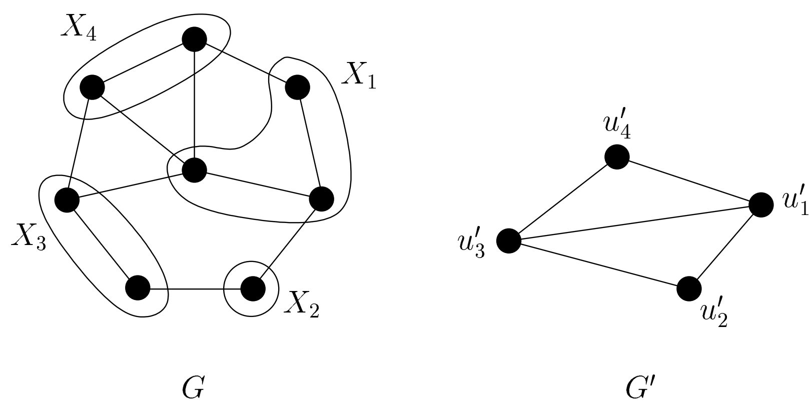

Figure 7.15: The graph G' is a contraction of G.

Observation 7.36
Consider the collection of all finite simple connected graphs $\mathcal{G}$. We have the following:

1. *Homeomorphism defines an equivalence relation $\approx$ on $\mathcal{G}$. That is, $G \approx G'$ if and only if G and G' are homeomorphic.*
2. *Contraction defines a partial order $\prec$ on $\mathcal{G}$. That is, $G' \prec G$ if and only if G' is a contraction of G.*

We consider the subdivision and contraction operations further. Let G be a simple connected graph that is homeomorphic to G', where G' has no vertex of degree two. In this case G' is not a subdivision of any graph, so G must itself be a subdivision of G'. In particular, G' must be a contraction of G. As we saw from Observation 7.28, G is planar if and only if G' is planar. However, this does not hold for contraction since a planar graph can be a contraction of a nonplanar graph. That is, K_4 is planar and a contraction of the nonplanar K_5.

The reversal of a subdivision is a sequence of contraction edges each with one endvertex of degree exactly two. Hence, contraction can be viewed as a more general operation.

➤ **Note 7.37**
If a graph G has a subgraph that is homeomorphic to a given graph H, then H is a contraction of that very subgraph. If, on the other hand, H is a minor of G, then it is not necessarily the case that G has a subgraph homeomorphic to H.

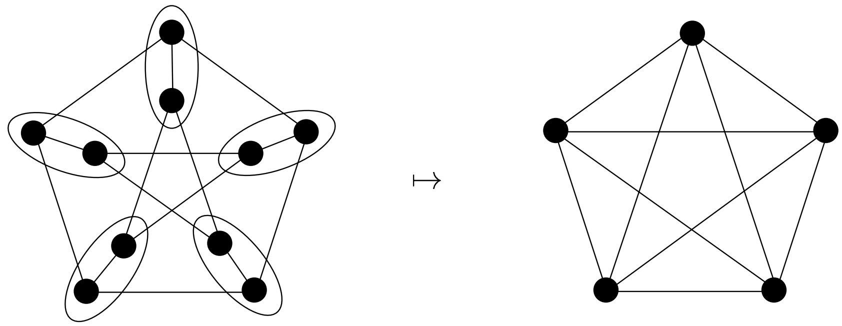

Figure 7.16: K_5 is a contraction of the Petersen graph.

The next example illustrates this.

☞ **Example 7.38**

Consider the Petersen graph. As we see in Figure 7.16, the Petersen graph has K_5 as a minor.

However, the Petersen graph has no subgraph that is homeomorphic to K_5 since all of the vertices of the Petersen graph have degree three. A graph that is homeomorphic to K_5 must have at least five vertices of degree four.

The fact that the Petersen graph has a subgraph homeomorphic to $K_{3,3}$ is not so clear from a first glance at the standard geometric representation of it. This can however be seen in the following way. The upper left part of Figure 7.17 shows the special geometric representation of the Petersen graph from Figure 2.7 in Chapter 2. By removing one vertex from the Petersen graph, we get a subgraph H that is homeomorphic to $K_{3,3}$. The fact that $K_{3,3}$ is homeomorphic to a subgraph of the Petersen graph implies by Kuratowski's Theorem 7.32 that the Petersen graph is not planar.

As we see from Example 7.38, contraction is "easier" to deal with than seeking a subgraph that is homeomorphic to some graph. It is precisely this generalization of Kuratowski's Theorem that was proved by Wagner in 1937. We present the theorem in the next section.

7.6 Kuratowski and Wagner's Theorem

The next theorem provides another classification of planar graphs.

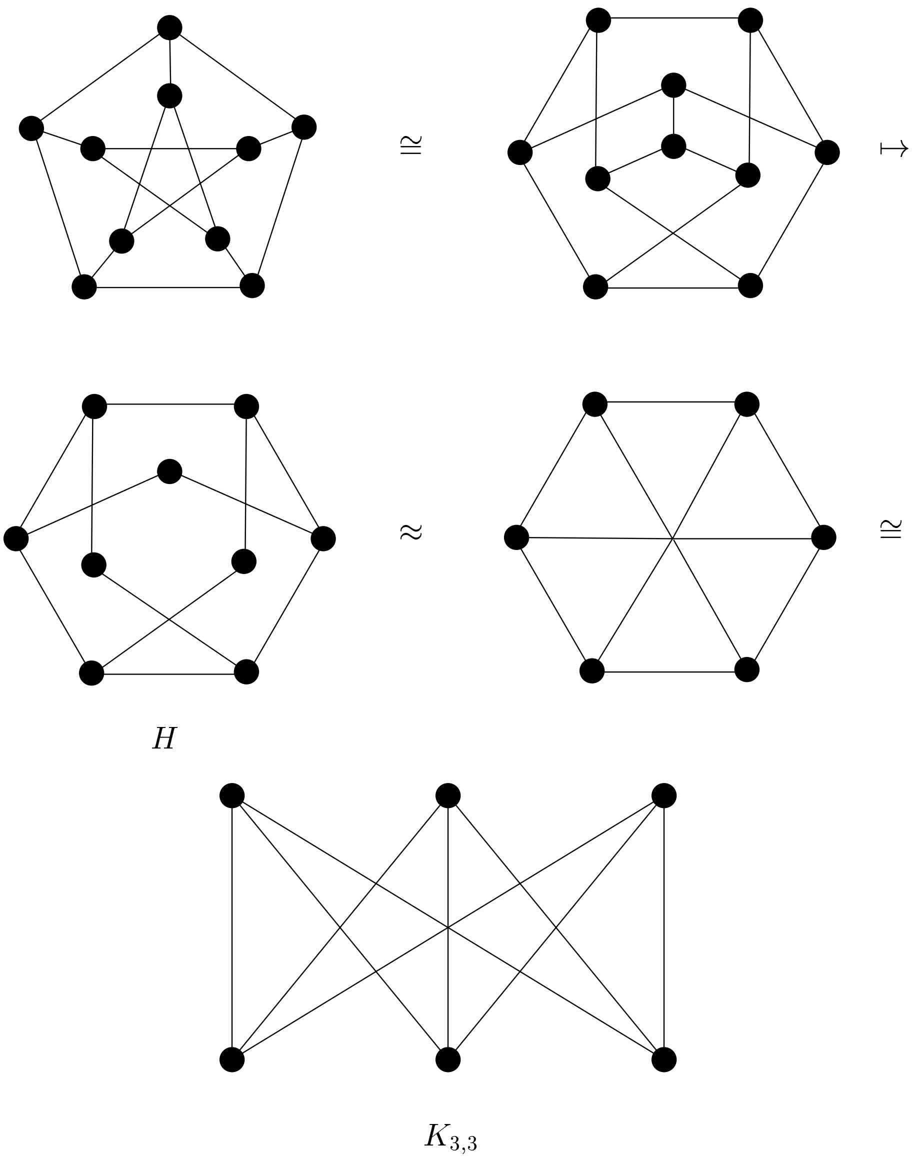

Figure 7.17: The Petersen graph has a subgraph H homeomorphic to $K_{3,3}$.

Theorem 7.39 (Kuratowski and Wagner's Theorem)
The following statements are equivalent for a graph G:

1. *The graph G is planar.*
2. *Neither K_5 nor $K_{3,3}$ are homeomorphic to any subgraph of G.*

3. *Neither K_5 nor $K_{3,3}$ are minors of G.*

PROOF: We sketch the proof of Wagner's part that the third condition is equivalent to the second condition.

If a subgraph H of G is homeomorphic to a given graph K with no vertices of degree two, then H must be a subdivision of K. Hence, K is a contraction of H. Since neither K_5 nor $K_{3,3}$ have any vertices of degree two, we have that condition three implies condition two.

To prove the other direction, we need to show that if either K_5 or $K_{3,3}$ is a minor of G, then G must contain a subgraph that is homeomorphic to either K_5 or $K_{3,3}$. We consider each case separately.

CASE ONE: $K_{3,3}$ is a contraction of a subgraph H of G. In this case there are six disjoint subsets H_1, H_2, H_3, K_1, K_2, and K_3 of $V(G)$ such that

- ▷ Each induces a connected simple subgraph in G, and
- ▷ For each $i, j \in \{1, 2, 3\}$ there is at least one edge in G with one endvertex $u_{i,j} \in H_i$ and the other $v_{i,j} \in K_j$.

Consider one of these sets, say H_i. This set contains the vertices $u_{i,1}, u_{i,2}$, and $u_{i,3}$ that are not necessarily distinct. Since the induced subgraph $G[H_i]$ connects all of these three vertices together, there is a subtree T_i of G with exactly three leaves $v_{1,i}$, $v_{2,i}$, and $v_{3,i}$ connecting all the vertices $u_{i,1}$, $u_{i,2}$, $u_{i,3}$, $v_{1,i}$, $v_{2,i}$, and $v_{3,i}$ together. Here the only vertices in T_i that are not in H_i are the $v_{i,l}$, where $l \in \{1, 2, 3\}$. It is easy to see that a tree with exactly three leaves is homeomorphic to $K_{1,3}$ as displayed in Figure 7.18, where a "zigzag-like" line indicates a simple path.

This also holds for all of the other sets K_j and the corresponding trees T'_j. Hence, G has a subgraph G' that is homeomorphic to $K_{3,3}$ and consists of the union of these six trees

$$G' = T_1 \cup T_2 \cup T_3 \cup T'_1 \cup T'_2 \cup T'_3.$$

CASE TWO: K_5 is a contraction of a subgraph H of G. In this case there are five disjoint subsets H_1, H_2, H_3, H_4, and H_5 of $V(G)$ such that each H_i induces a connected simple subgraph in G. Also, for each i not equal to j there

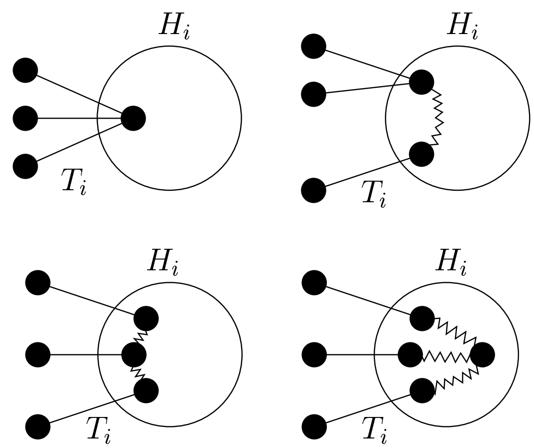

Figure 7.18: Each of these cases for T_i is homeomorphic to $K_{1,3}$.

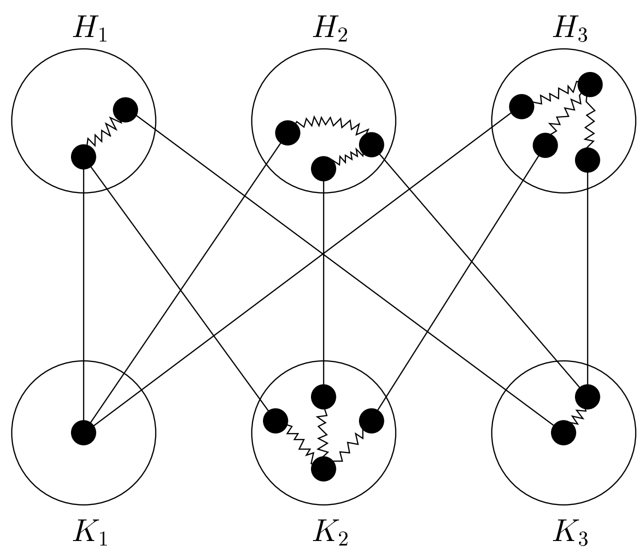

Figure 7.19: The subgraph G' of G is homeomorphic to $K_{3,3}$.

is an edge e_{ij} with endvertices $u_{i;\{i,j\}} \in H_i$ and $u_{j;\{i,j\}} \in H_j$. As in the first case, each H_i contains four vertices $u_{i;\{i,\alpha\}}, u_{i;\{i,\beta\}}, u_{i;\{i,\gamma\}}$, and $u_{i;\{i,\delta\}}$, where $\{\alpha, \beta, \gamma, \delta\} = \{1, 2, 3, 4, 5\} \setminus \{i\}$. Also, G has a subtree T_i with exactly four leaves $u_{\alpha;\{i,\alpha\}}, u_{\beta;\{i,\beta\}}, u_{\gamma;\{i,\gamma\}}$, and $u_{\delta;\{i,\delta\}}$ that connects all of the vertices $u_{i;\{i,l\}}$ and $u_{l,\{i,l\}}$ together, where $l \in \{\alpha, \beta, \gamma, \delta\}$. The only vertices not in H_i are the $u_{l;\{i,l\}}$. Figure 7.20 illustrates the situation.

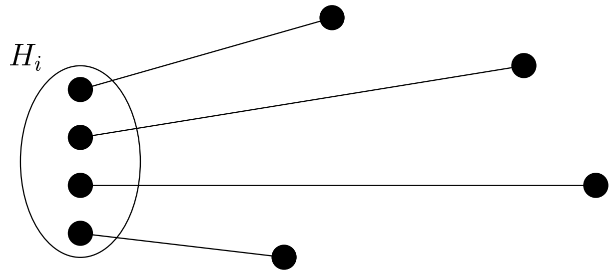

Figure 7.20: The subtree T_i of G connects the eight vertices shown.

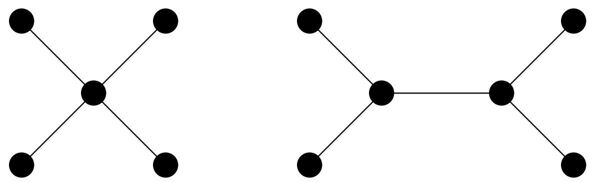

Figure 7.21: Each tree with exactly four leaves is homeomorphic to one of these trees.

The following lemma, which we need in this proof, is left to the reader in Exercise 12, and is illustrated in Figure 7.21.

Lemma 7.40
A tree with exactly four leaves is either homeomorphic to $K_{1,4}$ or the graph composed of an edge with two leaves connected to each endvertex of that edge.

Hence, T_i is homeomorphic to one of the two trees in Lemma 7.40. We need to consider two subcases.

SUBCASE ONE: If all of the trees T_i where $i \in \{1,2,3,4,5\}$ are homeomorphic to $K_{1,3}$, then G has a subgraph

$$G' = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$$

that is homeomorphic to K_5. This is illustrated in Figure 7.22.

SUBCASE TWO: There is at least one T_i that is homeomorphic to the second tree mentioned in Lemma 7.40. This tree consists of an edge with two leaves connected to each endvertex of the edge. Assume i equals one for the simplicity of indexing. Restricting our attention to T_1 and the edges e_{ij} together with their endvertices, where $i \in \{2,3\}$ and $j \in \{4,5\}$, we have the situation as displayed in Figure 7.23.

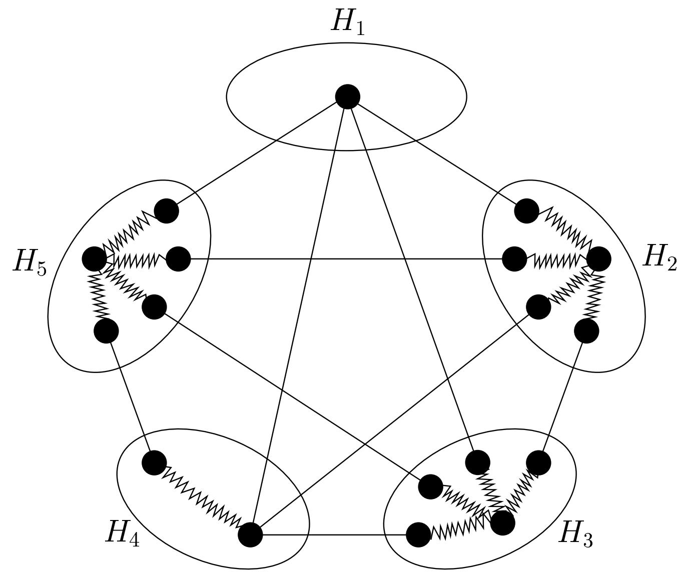

Figure 7.22: The graph G' is homeomorphic to K_5 since each T_i is homeomorphic to $K_{1,3}$.

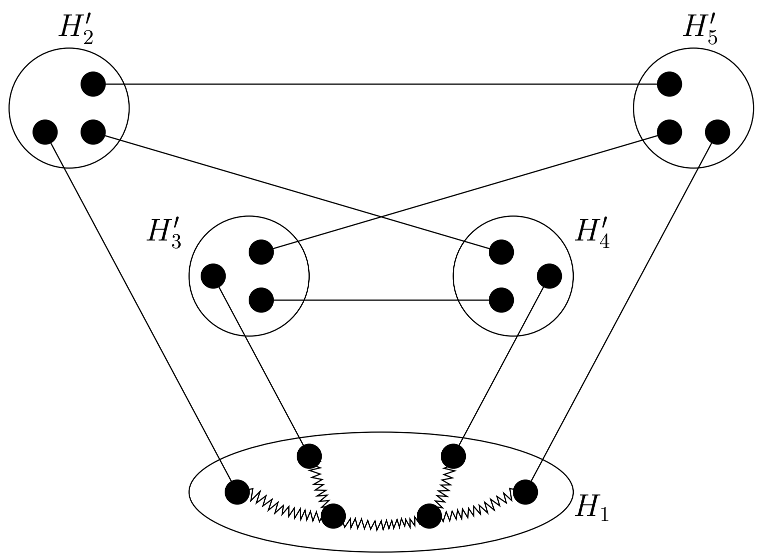

Figure 7.23: The tree T_1 together with the four sets of vertices H_2', H_3', H_4', and H_5'.

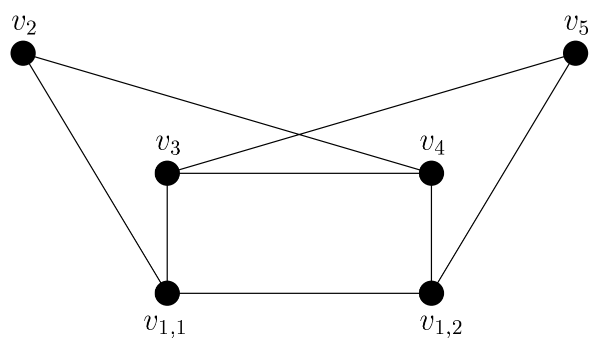

Figure 7.24: The subgraph G' of G is homeomorphic to this graph which is in turn isomorphic to $K_{3,3}$.

Here we are considering $H_i' \subseteq H_i$, where $i \in \{2,3,4,5\}$, each containing three vertices as listed.

$$\begin{aligned} H_2' &= \{u_{2;\{1,2\}}, u_{2;\{2,4\}}, u_{2;\{2,5\}}\}, \\ H_3' &= \{u_{3;\{1,3\}}, u_{3;\{3,4\}}, u_{3;\{3,5\}}\}, \\ H_4' &= \{u_{4;\{1,4\}}, u_{4;\{2,4\}}, u_{4;\{3,4\}}\}, \text{ and} \\ H_5' &= \{u_{5;\{1,5\}}, u_{5;\{2,5\}}, u_{5;\{3,5\}}\}. \end{aligned}$$

As in the first case, every tree with exactly three leaves is homeomorphic to $K_{1,3}$. Hence, G has a subgraph G' homeomorphic to the graph shown in Figure 7.24. This is isomorphic to $K_{3,3}$ since it has bipartition

$$\{v_2, v_3, v_{1,2}\} \cup \{v_4, v_5, v_{1,1}\}.$$

So, in this case G has a subgraph that is homeomorphic to $K_{3,3}$.

This completes the argument that the condition of Wagner implies that of Kuratowski, and we have the theorem. □

From this proof we extract the following corollary.

Corollary 7.41
Let G be a graph.

1. *If $K_{3,3}$ is a minor of G, then G contains a subgraph homeomorphic to $K_{3,3}$.*

2. *If K_5 is a minor of G, then G contains a subgraph homeomorphic to either K_5 or $K_{3,3}$.*

Note 7.42
Suppose n is greater than three. The graph $K_{1,n-1}$ used in the proof of Theorem 7.39 is a tree also called a star *on n vertices, or an* n-star.

Recall that if a graph G is planar, then so is every minor of G. A type of property closed under taking minors is sometimes called a *hereditary* property. Thus planarity is a hereditary property of graphs. By Theorem 7.39 planarity can be characterized as those graphs not having any of the graphs in the set $\{K_5, K_{3,3}\}$ as a minor. These are called the *forbidden* minors for planarity.

A deep theorem called the *Minor Theorem* by Robertson and Seymour, published as a series of many papers during the years from 1986 to 1997, implies that for *any* hereditary property P of graphs, there exists a finite collection $C(P)$ of graphs such that a graph G has property P if and only if G does not have any graphs in $C(P)$ as a minor. In particular, if P is the hereditary property of a graph being embeddable in a surface S other than the plane, we have the following corollary.

Corollary 7.43
For every surface S there exists a finite collection $\{H_1, \ldots, H_k\}$ of graphs such that a graph G is embeddable in S if and only if no H_i is a minor of G.

If the surface S is not the plane, then even to this day, the collections of forbidden minors corresponding to the surface S remains unknown. The only known set is the one corresponding to the plane given by Theorem 7.39. We discuss embeddings on other surfaces S in §8.6.

Although Theorem 7.39 gives an elegant and simple looking criterion for testing the planarity of a graph, the theorem is difficult to apply in the actual computational testing of a large graph. In particular, it is hard to apply on a large, simple, 2-connected graph, where each vertex has degree three or more. There are several alternative characterizations of a planar graph. We explore these in Chapter 8.

7.7 Exercises

1. Prove Euler's Formula in Theorem 7.19 inductively by using contraction of an edge in the graph.

2. Let G be a graph on n vertices, e edges, f faces, and with k components. Show that
$$n - e + f = k + 1.$$

3. Let G be a planar graph with no 3-cycles. Show that G has a vertex of degree less than four.

4. Show that every simple toroidal graph having seven or more vertices has a vertex of degree less than seven.

5. Assume that every face of a simple planar graph on n vertices and e edges, is bounded by at least k edges, show that
$$e \leq \frac{k(n-2)}{k-2}.$$

6. Use the following observation to show that K_5 is not planar.

   Observation 7.44
   Let C_4 be the four cycle embedded in the plane. If we connect each pair of opposite vertices with a continuous curve, drawing both in the same face of C_4, then these two curves must intersect.

7. Show that every face in a simple planar graph to which no edge can be added without destroying its planarity is a triangle.

8. Let G be a planar graph on n greater than three vertices. Show that G has at least four vertices of degree less than six.

9. A planar graph G is said to be *completely regular* if the degrees of all vertices of G are equal and every region is bounded by the same number of edges. Show that there are only five possible simple completely regular planar graphs, excluding the trivial graphs with vertices of degree less than three. These graph are sometimes called the graphs of the *Platonic solids*.

10. Prove that an infinite pattern formed of a regular polygon repeating itself, such as those found in mosaics and tiled floors, can consist of only three type of polygons—square, triangular, and hexagonal.

11. Redraw the graph shown in Figure 7.9 such that region two becomes the infinite region.

12. Let $i \in \{1, 2, 3\}$. Consider a simple graph G on six vertices u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, and u_{32}, where each pair of vertices is connected except for the three pairs $\{u_{i1}, u_{i2}\}$. Is G a planar graph?

13. By sketching all simple, nonseparable graphs with n less than five and e less than seven, prove that every planar graph not having a vertex of degree two is

    (a) a single edge together with both of its endvertices,

    (b) a complete graph on four vertices, or

    (c) a 2-connected graph on five or more vertices and seven or more edges.

14. Show that two graphs G' and G'' are homeomorphic if and only if there is a graph G''' that is a common subdivision of both of G' and G''.

15. Prove Observation 7.36.

16. Show that for two homeomorphic graphs G and G', we have

$$|V(G)| - |E(G)| = |V(G')| - |E(G')|.$$

17. Find an example of a planar graph G such that neither K_5 nor $K_{3,3}$ is a contraction of G. Find an example of a planar graph G that is neither homeomorphic to K_5 nor $K_{3,3}$. Why does this not contradict Theorem 7.39?

18. Let G be a graph such that for every two vertices u and v, there are at most two vertex-disjoint paths of length greater than one from u to v. Show that G is planar.

19. Show that if K_4 is a contraction of a graph G, then G has a subgraph that is homeomorphic to K_4. Does the same hold for K_5?

20. A graph G is called *outerplanar* if G has an embedding in the plane in such a way that each vertex bounds the infinite face. Prove the following Kuratowski-Wagner-like theorem.

    Theorem 7.45
    The following are equivalent for a graph G.

    (a) The graph G is outerplanar.
    (b) Neither K_4 nor $K_{2,3}$ are homeomorphic to any subgraph of G.
    (c) Neither K_4 nor $K_{2,3}$ are minors of G.