Freecalc
Homework on Lecture 3
Quiz time to be announced in class

1. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2 e^{-2x} \mathrm{d}x$. answer: $-\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

(b) $\int x \sin(2x) \mathrm{d}x$. answer: $-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

(c) $\int x \cos(3x) \mathrm{d}x$. answer: $\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$

(d) $\int x^2 e^{2x} \mathrm{d}x$ answer: $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C$

(e) $\int x^3 e^x \mathrm{d}x$ answer: $x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$

Solution. 1.a.

$$
\begin{aligned}
\int x^2 e^{-2x} \mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{Integrate by parts} \\
&= -\frac{x^2 e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right) \mathrm{d}\left(x^2\right) \\
&= -\frac{x^2 e^{-2x}}{2} + \int x e^{-2x} \mathrm{d}x \\
&= -\frac{x^2 e^{-2x}}{2} + \int x \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{Integrate by parts} \\
&= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} + \frac{1}{2}\int e^{-2x} \mathrm{d}x \\
&= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad .
\end{aligned}
$$

2. Evaluate the indefinite integral. Illustrate the steps of your solution.

(a) $\int x \sin(-2x) \mathrm{d}x$. answer: $\frac{x}{2}\cos(-2x) + \frac{1}{4}\sin(-2x) + C$

(b) $\int x^2 \cos(3x) \mathrm{d}x$. answer: $\frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$

(c) $\int x e^{-x} \mathrm{d}x$. answer: $-(1+x)e^{-x} + C$

3. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2 \cos(2x) \mathrm{d}x$.

answer: $\frac{1}{2}x^2 \sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$

(b) $\int x^2 e^{ax} \mathrm{d}x$, where a is a constant.

answer: $\frac{1}{a}x^2 e^{ax} - \frac{2}{a^2}x e^{ax} + \frac{2}{a^3}e^{ax} + C$

(c) $\int x^2 e^{-ax} \mathrm{d}x$, where a is a constant.

answer: $-\frac{1}{a}x^2 e^{-ax} - \frac{2}{a^2}x e^{-ax} - \frac{2}{a^3}e^{-ax} + C$

(d) $\int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} \mathrm{d}x$, where a is a constant.

answer: $\frac{1}{8}\left(a^{-1}x^2 e^{2ax} - a^{-1}x^2 e^{-2ax} - a^{-2}x e^{2ax} - a^{-2}x e^{-2ax} + \frac{1}{2}a^{-3}e^{2ax} - \frac{1}{2}a^{-3}e^{-2ax} + \frac{2}{3}x^3\right) + C$

(e) $\int e^{-\sqrt{x}} \mathrm{d}x$.

answer: $-2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

(f) $\int \cos^2 x \ \mathrm{d}x$

answer: $\frac{1}{4}\sin(2x) + \frac{x}{2} + C$

(g) $\int x \ln|x| \ \mathrm{d}x$

answer: $\frac{1}{2}x^2 \ln|x| - \frac{x^2}{4} + C$

(h) $\int \frac{x}{1+x^2} \mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

answer: $\frac{\ln(1+x^2)}{2} + C$

(i) $\displaystyle\int (\arctan x)\mathrm{d}x$

answer: $x \arctan x - \frac{\ln(1+x^2)}{2} + C$

(j) $\displaystyle\int (\arcsin x)\mathrm{d}x$

answer: $x \arcsin x + \sqrt{1-x^2} + C$

(k) $\displaystyle\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x$

answer: $2\sqrt{x}(\ln x - 2) + C$

(l) $\displaystyle\int (\arcsin x)^2\mathrm{d}x$ (Hint: Try substituting $x = \sin y$)

answer: $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(m) $\displaystyle\int \frac{1}{\cos^2 x}\mathrm{d}x$ (Hint: What is the derivative of $\tan x$?)

answer: $\tan x + C$

(n) $\displaystyle\int (\tan^2 x)\mathrm{d}x$ (Hint: $\tan^2 x = \frac{1}{\cos^2 x} - 1$)

(o) $\displaystyle\int x(\tan^2 x)\mathrm{d}x$ (Hint: $\tan^2 x\mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem)

(p) $\displaystyle\int \arctan\left(\frac{1}{x}\right)\mathrm{d}x$

(q) $\displaystyle\int \sin(\ln(x))\mathrm{d}x$

answer: $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$

(r) $\displaystyle\int \cos(\ln(x))\mathrm{d}x$

answer: $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$

(s) $\displaystyle\int (\ln x)^2\mathrm{d}x$.

(t) $\displaystyle\int (\ln x)^3\mathrm{d}x$.

(u) $\displaystyle\int x^2\cos^2 x\mathrm{d}x$ (This problem can be solved directly with integration by parts. An alternative and quicker solution is to use the fact that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and problem 3.d).

Solution. 3.e

$$\begin{aligned}
\int e^{-\sqrt{x}}\mathrm{d}x &= \int 2ye^{-y}\mathrm{d}y && \text{Subst.: } \sqrt{x} = y,\ \tfrac{1}{2\sqrt{x}}\mathrm{d}x = \mathrm{d}y,\ \mathrm{d}x = 2y\mathrm{d}y \\
&= \int 2y\mathrm{d}\left(-e^{-y}\right) && \text{int. by parts} \\
&= -2ye^{-y} + 2\int e^{-y}\mathrm{d}y \\
&= -2ye^{-y} - 2e^{-y} + C \\
&= -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C \quad .
\end{aligned}$$

Solution. 3.l.

$$\begin{aligned}
\int (\arcsin x)^2\mathrm{d}x &= \int (\arcsin(\sin y))^2\,\mathrm{d}(\sin y) && \text{Set } x = \sin y \\
&= \int y^2\cos y\mathrm{d}y = \int y^2\mathrm{d}(\sin y) && \text{Integrate by parts} \\
&= y^2\sin y - \int 2y\sin y\mathrm{d}y = y^2\sin y + \int 2y\mathrm{d}(\cos y) && \text{Integrate by parts} \\
&= y^2\sin y + 2y\cos y - 2\int \cos y\mathrm{d}y \\
&= y^2\sin y + 2y\cos y - 2\sin y + C && \text{Substitute back } y = \arcsin x \\
&= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C
\end{aligned}$$

Solution. 3.q

$$\begin{aligned}
\int \sin(\ln x)\mathrm{d}x &= x\sin(\ln x) - \int x\mathrm{d}(\sin(\ln x)) && \Big|\ \text{integrate by parts}\\
&= x\sin(\ln x) - \int x\left(\cos(\ln x)\right)(\ln x)'\,\mathrm{d}x\\
&= x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x && \Big|\ \text{integrate by parts again}\\
&= x\sin(\ln x) - \left(x\cos(\ln x) - \int x\mathrm{d}(\cos(\ln x))\right)\\
&= x\sin(\ln x) - x\cos(\ln x) + \int x(-\sin(\ln x))(\ln x)'\mathrm{d}x\\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x && \Big|\ \text{add } \textstyle\int \sin(\ln x)\mathrm{d}x \text{ to both sides}\\
2\int \sin(\ln x)\mathrm{d}x &= x\sin(\ln x) - x\cos(\ln x)\\
\int \sin(\ln x)\mathrm{d}x &= \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right)
\end{aligned}$$