Freecalc
Homework on Lecture 7
Will be quizzed: date to be announced

1. Compute the integral using a trigonometric substitution.

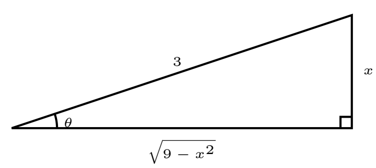
   (a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

answer: $-\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C$

Solution. 1.a

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}(3\cos\theta)\mathrm{d}\theta \\ &= 9\int\frac{|\cos\theta|}{\sin^2\theta}\cos\theta\mathrm{d}\theta \\ &= \int \cot^2\theta\mathrm{d}\theta \\ &= \int (csc^2\theta - 1)\mathrm{d}\theta \\ &= -\cot\theta - \theta + C \\ &= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C,\end{aligned}$$

Set $x = 3\sin\theta$
for $\theta \in \left[\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\mathrm{d}x = 3\cos\theta\mathrm{d}\theta$
For $\theta \in \left[\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
we have $|\cos\theta| = \cos\theta$



where we expressed $\cot\theta$ via $\sin\theta$ by considering the following triangle.

2. (a) In the Euler substitution $x = \cot(2\arctan t)$, express $x, \sqrt{x^2+1}$ via t, $\mathrm{d}x$ via $\mathrm{d}t$, and t via x.

   (b) Integrate

   i. $$\int\sqrt{x^2+1}\mathrm{d}x$$

   ii. $$\int\sqrt{x^2+x+1}\mathrm{d}x$$

   iii. $$\int\sqrt{x^2+2}\mathrm{d}x$$

   iv. $$\int\sqrt{(2x^2+2x+1)}\mathrm{d}x$$

answer: $\frac{\sqrt{2}}{4}\left(\frac{1}{2}(2x+1)\sqrt{(2x+1)^2+1} + \frac{1}{2}\ln\left(\sqrt{(2x+1)^2+1} + 2x+1\right)\right) + C$

Solution. 2(2.b)iv

$$\begin{aligned}
\int \sqrt{(2x^2+2x+1)}\mathrm{d}x &= \int \sqrt{2}\sqrt{\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)}\mathrm{d}x && \text{complete the square}\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left(4\left(x+\frac{1}{2}\right)^2+1\right)}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left((2x+1)^2+1\right)}\frac{1}{2}\mathrm{d}\left(2x+1\right) && \text{Set } u=2x+1\\
& && \text{Use Euler substitution:}\\
& && u=\tfrac{1}{2}\left(t^{-1}-t\right),\quad t>0\\
&= \frac{\sqrt{2}}{4}\int \sqrt{(u^2+1)}\mathrm{d}u && \mathrm{d}u=-\tfrac{1}{2}\left(t^{-2}+1\right)\mathrm{d}t\\
& && \sqrt{u^2+1}=\tfrac{1}{2}\left(t^{-1}+t\right)\\
& && t=\sqrt{u^2+1}-u\\
&= -\frac{\sqrt{2}}{16}\int \left(t^{-1}+t\right)\left(t^{-2}+1\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\int \left(t^{-3}+2t^{-1}+t\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&= \frac{\sqrt{2}}{4}\left(-\frac{1}{4}\left(-\frac{\left(\sqrt{u^2+1}-u\right)^{-2}}{2}\right.\right.\\
&\qquad \left.\left.+\ 2\ln\left(\sqrt{u^2+1}-u\right)+\frac{\left(\sqrt{u^2+1}-u\right)^2}{2}\right)\right)+C && \text{simplifies according to theory}\\
&= \frac{\sqrt{2}}{4}\left(\frac{1}{2}u\sqrt{u^2+1}+\frac{1}{2}\ln\left(\sqrt{u^2+1}+u\right)\right)+C\\
&= \frac{\sqrt{2}}{4}\left(\frac{1}{2}(2x+1)\sqrt{(2x+1)^2+1}\right.\\
&\qquad \left.+\frac{1}{2}\ln\left(\sqrt{(2x+1)^2+1}+2x+1\right)\right)+C
\end{aligned}$$

3. (a) In the Euler substitution $x=\cos(2\arctan t)$, express $x, \sqrt{1-x^2}$ via t, $\mathrm{d}x$ via $\mathrm{d}t$ and t via x.
   (b) Integrate
      i.
      $$\int \sqrt{1-x^2}\mathrm{d}x$$
      ii.
      $$\int \sqrt{2-x^2}\mathrm{d}x$$
      iii.
      $$\int \sqrt{-x^2+x+1}\mathrm{d}x$$
4. (a) In the Euler substitution $x=\sec(2\arctan t)$, express $x, \sqrt{x^2-1}$ via t, dx via dt and t via x.
   (b) Integrate
      i.
      $$\int \sqrt{x^2-1}\mathrm{d}x$$
      ii.
      $$\int \sqrt{x^2-2}\mathrm{d}x$$
      iii.
      $$\int \sqrt{x^2+x-1}\mathrm{d}x$$