Math 141
Lecture 5

Greg Maloney

Todor Milev

University of Massachusetts Boston

Spring 2015

Outline

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} =$$

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} =$$

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

From building blocks to all rational functions: example

- We know how to solve $\displaystyle\int \frac{2}{x-1}\mathrm{d}x$ and $\displaystyle\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x =$$

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x =$$

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x = 2\ln|x-1| - \ln|x+2| + C$$

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1} \mathrm{d}x$ and $\int \frac{1}{x+2} \mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2} \mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) \mathrm{d}x = 2 \ln |x-1| - \ln |x+2| + C$$

- From (linear substitutions of) basic building blocks we constructed a larger example, which we can therefore solve.

From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x = 2\ln|x-1| - \ln|x+2| + C$$

- From (linear substitutions of) basic building blocks we constructed a larger example, which we can therefore solve.
- We will now learn how to do the reverse procedure: given a rational function, split it into "partial fractions" which are transformed by linear substitutions to basic building block integrals.

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

- We already learned know how to integrate all partial fractions (using linear substitutions and building blocks I, II and III).

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

- We already learned know how to integrate all partial fractions (using linear substitutions and building blocks I, II and III).
- Thus, if we can produce the partial fractions whose existence is promised by the theorem, we can integrate all rational functions.

Review of polynomial notation

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. Recall that the degree of P is the highest power of x in P that has a non-zero coefficient. That is, if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n, and we write $\deg(P) = n$.

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and

$$P(x) = S(x)Q(x) + R(x)$$

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and
  $$P(x) = S(x)Q(x) + R(x)$$

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and
  $$P(x) = S(x)Q(x) + R(x)$$

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and
  $$P(x) = S(x)Q(x) + R(x)$$

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and
  $$P(x) = S(x)Q(x) + R(x)$$

Ensure denominator degree > numerator degree

- To compute a partial fraction decomposition we need that the degree of the fraction numerator be less than the degree of the denominator.
- Therefore our first step is to transform $\frac{P(x)}{Q(x)}$ to
  $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$
  where $S(x), R(x), Q(x)$ are polynomials and $\deg R < \deg Q$.
- This is done using polynomial long division.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and

$$P(x) = S(x)Q(x) + R(x)$$

- We review polynomial long division on examples.

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$x - 1 \overline{\big) x^3 \qquad + \ x}$$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$x-1 \overline{\big) x^3 \qquad + \; x}$$

Divide x^3 by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$\begin{array}{r} x^2 \\ x-1 \overline{) x^3 \quad + \quad x} \end{array}$$

Divide x^3 by x

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$\begin{array}{r|l} & x^2 \\ \hline x-1 & x^3 \quad + \; x \end{array}$$

Multiply x^2 by $x-1$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$\begin{array}{r|l} & x^2 \\ \hline x-1 & x^3 \quad\quad + \; x \\ & \underline{x^3 - x^2} \end{array}$$

Multiply x^2 by $x-1$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$\begin{array}{r|l} & x^2 \\ x-1 & x^3 \quad + \quad x \\ & x^3 - x^2 \end{array}$$

Subtract $x^3 - x^2$ from x^3

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 \\
x - 1 & x^3 \qquad + \; x \\
 & x^3 - x^2 \\
 & \qquad x^2
\end{array}
$$

Subtract $x^3 - x^2$ from x^3

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r}
x^2 \\
x-1 \overline{\big) x^3 + x} \\
\underline{x^3 - x^2} \\
x^2 + x
\end{array}
$$

Bring down the x

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{rl}
 & x^2 \\
x-1 \big) & x^3 \qquad + x \\
 & x^3 - x^2 \\
\hline
 & \qquad x^2 + x
\end{array}
$$

Divide x^2 by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$\begin{array}{r|l} & x^2 + x \\ \hline x - 1 & x^3 \quad\quad + x \\ & x^3 - x^2 \\ \hline & \quad\quad x^2 + x \\ \hline \end{array}$$

Divide x^2 by x

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-1 & x^3 \quad + x \\
 & x^3 - x^2 \\
\hline
 & \quad x^2 + x
\end{array}
$$

Multiply x by $x-1$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$\begin{array}{r|l} & x^2 + x \\ \hline x-1 & x^3 \qquad + x \\ & x^3 - x^2 \\ \hline & \qquad x^2 + x \\ & \qquad x^2 - x \\ \hline \end{array}$$

Multiply x by $x-1$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x - 1 & x^3 \quad + x \\
 & \underline{x^3 - x^2} \\
 & \quad x^2 + x \\
 & \quad \underline{x^2 - x}
\end{array}
$$

Subtract $x^2 - x$ from $x^2 + x$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x \\
x - 1 & x^3 \qquad + x \\
 & x^3 - x^2 \\
 & \quad x^2 + x \\
 & \quad x^2 - x \\
 & \qquad 2x
\end{array}
$$

Subtract $x^2 - x$ from $x^2 + x$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x - 1 & x^3 \quad\quad + x \\
 & x^3 - x^2 \\
\hline
 & \quad\quad x^2 + x \\
 & \quad\quad x^2 - x \\
\hline
 & \quad\quad\quad\quad 2x
\end{array}
$$

Divide $2x$ by x

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 \quad\quad + x \\
 & x^3 - x^2 \\
\hline
 & \quad\; x^2 + x \\
 & \quad\; x^2 - x \\
\hline
 & \quad\quad\quad 2x
\end{array}
$$

Divide $2x$ by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
x - 1 & x^3 \quad\quad + x \\
 & x^3 - x^2 \\
\hline
 & \quad x^2 + x \\
 & \quad x^2 - x \\
\hline
 & \quad\quad 2x
\end{array}
$$

Multiply 2 by $x - 1$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 \qquad + x \\
 & x^3 - x^2 \\
\hline
 & \qquad x^2 + x \\
 & \qquad x^2 - x \\
\hline
 & \qquad\qquad 2x \\
 & \qquad\qquad 2x - 2 \\
\hline
\end{array}
$$

Multiply 2 by $x - 1$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 \qquad + x \\
 & x^3 - x^2 \\
 & \qquad x^2 + x \\
 & \qquad x^2 - x \\
 & \qquad\qquad 2x \\
 & \qquad\qquad 2x - 2
\end{array}
$$

Subtract $2x-2$ from $2x$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x-1 & x^3 \qquad + x \\ & x^3 - x^2 \\ \hline & \qquad x^2 + x \\ & \qquad x^2 - x \\ \hline & \qquad\quad 2x \\ & \qquad\quad 2x - 2 \\ \hline & \qquad\qquad 2 \end{array}$$

Subtract $2x-2$ from $2x$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{rl}
 & x^2 + x + 2 \\
x - 1 \big) & x^3 \qquad + x \\
 & x^3 - x^2 \\
 & \qquad x^2 + x \\
 & \qquad x^2 - x \\
 & \qquad\qquad 2x \\
 & \qquad\qquad 2x - 2 \\
 & \qquad\qquad\quad 2
\end{array}
$$

$$
\begin{aligned}
& \int \frac{x^3+x}{x-1} dx \\
= & \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) dx
\end{aligned}
$$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$\begin{array}{r} x^2 + x + 2 \\ x-1 \overline{) x^3 \qquad + x} \\ \underline{x^3 - x^2} \qquad \\ x^2 + x \\ \underline{x^2 - x} \\ 2x \\ \underline{2x - 2} \\ 2 \end{array}$$

$$\int \frac{x^3+x}{x-1} dx = \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) dx$$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
x-1 & x^3 \qquad + x \\
 & x^3 - x^2 \\
 & \quad x^2 + x \\
 & \quad x^2 - x \\
 & \qquad 2x \\
 & \qquad 2x - 2 \\
 & \qquad\quad 2
\end{array}
$$

$$
\int \frac{x^3+x}{x-1}\mathrm{d}x = \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) \mathrm{d}x
$$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r}
x^2 + x + 2 \\
x-1 \overline{\big) x^3 \qquad + x} \\
\underline{x^3 - x^2} \qquad \\
x^2 + x \\
\underline{x^2 - x} \\
2x \qquad \\
\underline{2x - 2} \\
2
\end{array}
$$

$$
\begin{aligned}
& \int \frac{x^3+x}{x-1}\mathrm{d}x \\
= & \int \left(x^2+x+2+\frac{2}{x-1}\right)\mathrm{d}x \\
= & \frac{x^3}{3}+\frac{x^2}{2}+2x \\
& +2\ln|x-1|+C
\end{aligned}
$$

- The next step in producing a partial fraction decomposition is to factor the denominator $Q(x)$.
- Factoring of $Q(x)$ can always be done in quadratic and linear terms:

Corollary (Corollary to the Fundamental Theorem of Algebra)

Let $Q(x)$ be a polynomial (with real coefficients). Then $Q(x)$ can be factored as a product of terms of the form $(ax+b)^n$ (powers of linear terms) and product of terms of the form $(ax^2+bx+c)^n$ with $b^2-4ac<0$ (powers of quadratic terms).

- The above result is a corollary to the Fundamental Theorem of Algebra. We state the Fundamental Theorem of algebra without proving it.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial has at least one complex root.

Suppose we have already factored the denominator $Q(x)$ into factors of the form

$$(ax+b)^N \qquad \text{and} \qquad (ax^2+bx+c)^N$$

Suppose we have already factored the denominator $Q(x)$ into factors of the form

$$(ax+b)^N \qquad \text{and} \qquad (ax^2+bx+c)^N$$

Then we can split the fraction $R(x)/Q(x)$ into sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^i} \quad ,$$

where the exponent i in the partial fraction does not exceed the exponent N of the corresponding term in $Q(x)$.

Suppose we have already factored the denominator $Q(x)$ into factors of the form

$$(ax+b)^N \qquad \text{and} \qquad (ax^2+bx+c)^N$$

Then we can split the fraction $R(x)/Q(x)$ into sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^i} \quad ,$$

where the exponent i in the partial fraction does not exceed the exponent N of the corresponding term in $Q(x)$.
The cases when the factorization of $Q(x)$ has terms appearing with power $N > 1$ are treated differently from the the cases where all terms of the factorization of $Q(x)$ are distinct.

Suppose $Q(x)$ is a product of distinct linear factors.
This means we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).
Then there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

The next example shows how to find $A_1, A_2, \cdots, A_k$.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2 + 2x - 1) < \deg(2x^3 + 3x^2 - 2x)$: don't divide.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2 + 2x - 1) < \deg(2x^3 + 3x^2 - 2x)$: don't divide.
- Factor denominator: $2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$.

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\ x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\end{aligned}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\ x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\ x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A\end{aligned}$$

$$2A+B+2C = 1$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\ x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\ x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A\end{aligned}$$

$$\begin{aligned}2A+\ \ B+2C &= 1\\ 3A+2B-\ \ C &= 2\end{aligned}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\
x^2+2x-1 &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A
\end{aligned}$$

$$\begin{aligned}
2A + B + 2C &= 1 \\
3A + 2B - C &= 2 \\
-2A &= -1
\end{aligned}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\
x^2+2x-1 &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A
\end{aligned}$$

$$\begin{aligned}
2A + B + 2C &= 1 \\
3A + 2B - C &= 2 \\
-2A &= -1
\end{aligned}$$

Solution:
$A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

$$\begin{aligned}
2A+B+2C &= 1\\
3A+2B-C &= 2\\
-2A &= -1
\end{aligned}$$

Solution:
$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x = \int\left(\frac{1}{2}\,\frac{1}{x}+\frac{1}{5}\,\frac{1}{2x-1}-\frac{1}{10}\,\frac{1}{x+2}\right)\mathrm{d}x$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

$$\begin{aligned}
2A+B+2C &= 1\\
3A+2B-C &= 2\\
-2A &= -1
\end{aligned}$$

Solution:
$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x = \int\left(\frac{1}{2}\,\frac{1}{x}+\frac{1}{5}\,\frac{1}{2x-1}-\frac{1}{10}\,\frac{1}{x+2}\right)\mathrm{d}x$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned} \frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\ x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\ x^2+2x-1 &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A \end{aligned}$$

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

Solution:

$A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$.

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x = \int \left(\frac{1}{2}\,\frac{1}{x} + \frac{1}{5}\,\frac{1}{2x-1} - \frac{1}{10}\,\frac{1}{x+2} \right) \mathrm{d}x$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2 + 2x - 1) < \deg(2x^3 + 3x^2 - 2x)$: don't divide.
- Factor denominator: $2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$.

$$\begin{aligned}
\frac{x^2 + 2x - 1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
x^2 + 2x - 1 &= A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1) \\
x^2 + 2x - 1 &= (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A
\end{aligned}$$

$$\begin{aligned}
2A + B + 2C &= 1 \\
3A + 2B - C &= 2 \\
-2A &= -1
\end{aligned}$$

Solution:
$A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$.

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x \\
= &\int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2} \right) \mathrm{d}x
\end{aligned}$$

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

$$\begin{aligned}
2A+\ B+2C &= 1\\
3A+2B-\ C &= 2\\
-2A\qquad &= -1
\end{aligned}$$

Solution:
$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x\\
&= \int\left(\frac{1}{2}\,\frac{1}{x}+\frac{1}{5}\,\frac{1}{2x-1}-\frac{1}{10}\,\frac{1}{x+2}\right)\mathrm{d}x\\
&= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|\\
&\qquad -\frac{1}{10}\ln|x+2|+K
\end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$;

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$;

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$0^2+2\cdot 0-1 \quad = \quad A(2\cdot 0-1)(0+2)$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{aligned} 0^2 + 2 \cdot 0 - 1 &= A(2 \cdot 0 - 1)(0 + 2) \\ -1 &= -2A \end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{aligned} 0^2 + 2 \cdot 0 - 1 &= A(2 \cdot 0 - 1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$\begin{aligned} 0^2+2\cdot 0-1 &= A(2\cdot 0-1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}-1 = B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{aligned} 0^2 + 2\cdot 0 - 1 &= A(2\cdot 0 - 1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2} - 1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &= \frac{5}{4}B \end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$\begin{aligned} 0^2+2\cdot 0-1 &= A(2\cdot 0-1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}-1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &= \frac{5}{4}B \\ B &= \frac{1}{5} \end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{aligned} 0^2 + 2\cdot 0 - 1 &= A(2\cdot 0 - 1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2} - 1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &= \frac{5}{4}B \\ B &= \frac{1}{5} \end{aligned}$$

$$(-2)^2 + 2(-2) - 1 = C(-2)(2(-2)-1)$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$\begin{aligned} 0^2+2\cdot 0-1 &= A(2\cdot 0-1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}-1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &= \frac{5}{4}B \\ B &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} (-2)^2+2(-2)-1 &= C(-2)(2(-2)-1) \\ -1 &= 10C \end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{aligned} 0^2 + 2\cdot 0 - 1 &= A(2\cdot 0 - 1)(0+2) \\ -1 &= -2A \\ A &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2} - 1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &= \frac{5}{4}B \\ B &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} (-2)^2 + 2(-2) - 1 &= C(-2)(2(-2)-1) \\ -1 &= 10C \\ C &= -\frac{1}{10} \end{aligned}$$

Suppose $Q(x)$ is a product of linear factors, some of which appear with power greater than 1.
Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of a single term $A/(a_1x + b_1)$ we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

We make similar adjustments for all other repeating terms $(a_sx + b_s)$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\mathrm{d}x$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

$$\begin{aligned} \frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\ 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \end{aligned}$$

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\mathrm{d}x$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1=(x-1)^2(x+1)$.

$$\begin{aligned}\frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$

- Plug in -1:

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.

$$\begin{aligned}\frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C=-1$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.

$$\begin{aligned} \frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1} \\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2 \end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C=-1$.
- Plug in 1:

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.

$$\begin{aligned}\frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C=-1$.
- Plug in 1: $4(1) = B(1+1)$ therefore $B=2$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

$$\begin{aligned} \frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\ 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C = -1$.
- Plug in 1: $4(1) = B(1+1)$ therefore $B = 2$.
- Plug in 0: $4(0) = A(0-1)(0+1) + 2(0+1) + (-1)(0-1)^2$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

$$\begin{aligned} \frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\ 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C = -1$.
- Plug in 1: $4(1) = B(1+1)$ therefore $B = 2$.
- Plug in 0: $4(0) = A(0-1)(0+1) + 2(0+1) + (-1)(0-1)^2$.
- Therefore $A = 1$.

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\mathrm{d}x$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1=(x-1)^2(x+1)$.

$$\begin{aligned}\frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$

- Plug in -1: $4(-1)=C(-1-1)^2$, therefore $C=-1$.
- Plug in 1: $4(1)=B(1+1)$ therefore $B=2$.
- Plug in 0: $4(0)=A(0-1)(0+1)+2(0+1)+(-1)(0-1)^2$.
- Therefore $A=1$.

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\mathrm{d}x = \int\left(x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right)\mathrm{d}x$$

Example (Example 4, p. 513)

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$.

- Divide: $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.

$$\begin{aligned}\frac{4x}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$

- Plug in -1: $4(-1) = C(-1-1)^2$, therefore $C=-1$.
- Plug in 1: $4(1) = B(1+1)$ therefore $B = 2$.
- Plug in 0: $4(0) = A(0-1)(0+1)+2(0+1)+(-1)(0-1)^2$.
- Therefore $A = 1$.

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx &= \int\left(x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right)dx\\ &= \frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+K\end{aligned}$$

Suppose $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions arising from linear factors, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

This term can be integrated by completing the square and using the formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

Example

Find $\int \frac{2x^2-x+4}{x^3+4x} dx$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{(x^2+4)}$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned} \frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\ 2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \end{aligned}$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A + B)x^2 + Cx + 4A
\end{aligned}$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A + B)x^2 + Cx + 4A
\end{aligned}$$

$A = 1$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x} \mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$
\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A+B)x^2 + Cx + 4A
\end{aligned}
$$

$A = 1 \qquad C = -1$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x} \mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$
\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2 + 4)} \\
2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\
2x^2 - x + 4 &= (A + B)x^2 + Cx + 4A
\end{aligned}
$$

$$A = 1 \qquad C = -1 \qquad A + B = 2$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)}\\ 2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x\\ 2x^2 - x + 4 &= (A+B)x^2 + Cx + 4A\end{aligned}$$

$A = 1 \qquad C = -1 \qquad A + B = 2,$ therefore $B = 1$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x} \mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A + B)x^2 + Cx + 4A
\end{aligned}$$

$$A = 1 \qquad C = -1 \qquad A + B = 2, \text{ therefore } B = 1$$

$$\int \frac{2x^2 - x + 4}{x(x^2+4)} \mathrm{d}x = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) \mathrm{d}x$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2-x+4) < \deg(x^3+4x)$: don't divide.
- Factor denominator: $x^3+4x = x(x^2+4)$.

$$\begin{aligned}
\frac{2x^2-x+4}{x(x^2+4)} &= \frac{A}{x}+\frac{Bx+C}{(x^2+4)}\\
2x^2-x+4 &= A(x^2+4)+(Bx+C)x\\
2x^2-x+4 &= (A+B)x^2+Cx+4A
\end{aligned}$$

$$A=1 \qquad C=-1 \qquad A+B=2, \text{ therefore } B=1$$

$$\begin{aligned}
\int \frac{2x^2-x+4}{x(x^2+4)}\mathrm{d}x &= \int\left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)\mathrm{d}x\\
&= \int\frac{1}{x}\mathrm{d}x+\int\frac{x}{x^2+4}\mathrm{d}x-\int\frac{1}{x^2+4}\mathrm{d}x
\end{aligned}$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A+B)x^2 + Cx + 4A
\end{aligned}$$

$$A = 1 \qquad C = -1 \qquad A + B = 2, \text{ therefore } B = 1$$

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x(x^2+4)}\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)\mathrm{d}x \\
&= \int \frac{1}{x}\mathrm{d}x + \int \frac{x}{x^2+4}\mathrm{d}x - \int \frac{1}{x^2+4}\mathrm{d}x \\
&= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K
\end{aligned}$$

Suppose $Q(x)$ contains irreducible quadratic factors, some of which are repeated.
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single term $(Ax + B)/(ax^2 + bx + c)$ we use

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

in the partial fraction decomposition of $R(x)/Q(x)$.
These terms can be integrated by completing the square.

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} +$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$