Math 141

Lecture 4

Greg Maloney

Todor Milev

University of Massachusetts Boston

Spring 2015

Outline

1. Integration of Rational Functions
   - Building block integrals

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} dx$*?*

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces ("partial fractions").

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces ("partial fractions").
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\displaystyle\int \frac{P(x)}{Q(x)} dx$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces (“partial fractions”).
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.

Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\displaystyle\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces ("partial fractions").
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.
- We study the algorithm "from the ground up": we start with the building blocks.

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n} \mathrm{d}x \quad .$$

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n} \mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
$$\int \frac{x}{(1+x^2)^n} \mathrm{d}x.$$

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
  $$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
  $\int \frac{x}{(1+x^2)^n}\mathrm{d}x$. (Note: $u = 1 + x^2$, $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
$\int \frac{x}{(1+x^2)^n}\mathrm{d}x$. (Note: $u = 1 + x^2$, $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
  $$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
  $\int \frac{x}{(1+x^2)^n}\mathrm{d}x$. (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
  $$\int \frac{1}{(1+x^2)^n}\mathrm{d}x \quad .$$

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
  $$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
  $\displaystyle\int \frac{x}{(1+x^2)^n}\mathrm{d}x.$ (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
  $$\int \frac{1}{(1+x^2)^n}\mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
  $\displaystyle\int \frac{1}{x}\mathrm{d}x, \int \frac{x}{1+x^2}\mathrm{d}x$ and $\displaystyle\int \frac{1}{1+x^2}\mathrm{d}x.$

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
  $$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
  $\displaystyle\int \frac{x}{(1+x^2)^n}\mathrm{d}x$. (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
  $$\int \frac{1}{(1+x^2)^n}\mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
  $\displaystyle\int \frac{1}{x}\mathrm{d}x, \int \frac{x}{1+x^2}\mathrm{d}x$ and $\displaystyle\int \frac{1}{1+x^2}\mathrm{d}x$.
- The case $n = 1$ we call respectively building block Ia, IIa and IIIa.

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
  $$\int \frac{1}{x^n} \mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
  $$\int \frac{x}{(1+x^2)^n} \mathrm{d}x.$$ (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
  $$\int \frac{1}{(1+x^2)^n} \mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
  $$\int \frac{1}{x} \mathrm{d}x, \int \frac{x}{1+x^2} \mathrm{d}x \text{ and } \int \frac{1}{1+x^2} \mathrm{d}x.$$
- The case $n = 1$ we call respectively building block Ia, IIa and IIIa. The case $n > 1$ we call respectively building block Ib, IIb and IIIb.

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
$\int \frac{x}{(1+x^2)^n}\mathrm{d}x$. (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
$$\int \frac{1}{(1+x^2)^n}\mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
$\int \frac{1}{x}\mathrm{d}x$, $\int \frac{x}{1+x^2}\mathrm{d}x$ and $\int \frac{1}{1+x^2}\mathrm{d}x$.
- The case $n = 1$ we call respectively building block Ia, IIa and IIIa. The case $n > 1$ we call respectively building block Ib, IIb and IIIb. This "building block" terminology serves our convenience, and is not a part of standard mathematical terminology.

Building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x$.

Example

Integrate building block Ia

$$\int \frac{1}{x} \mathrm{d}x$$

Building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x$.

Example

Integrate building block Ia

$$\int \frac{1}{x} \mathrm{d}x =$$

Building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x$.

Example

Integrate building block Ia

$$\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x = \ln |x| + C$.

Example

Integrate

$$\int \frac{1}{-4x+5} \mathrm{d}x$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$.

Example

Integrate

$$\int \frac{1}{-4x+5}\mathrm{d}x = \int \frac{1}{(-4x+5)} \frac{\mathrm{d}(\,-4x)}{(\,-4)}$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\int \frac{1}{-4x+5}\mathrm{d}x = \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)}$$
$$= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)}$$

Linear substitutions leading to building block Ia

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)}\end{aligned}$$

Set $u = -4x + 5$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)} \frac{\mathrm{d}(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)} \frac{\mathrm{d}(-4x+5)}{(-4)} \\ &= \int \frac{1}{u} \frac{\mathrm{d}u}{(-4)} \\ &= -\frac{1}{4} \int u^{-1} \mathrm{d}u\end{aligned}$$

Set $u = -4x + 5$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)}\\ &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} \qquad \Big| \text{ Set } u = -4x+5\\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)}\\ &= -\frac{1}{4}\int u^{-1}\mathrm{d}u\end{aligned}$$

Linear substitutions leading to building block Ia

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)} \\ &= -\frac{1}{4}\int u^{-1}\mathrm{d}u = -\frac{1}{4}\ln|u| + C\end{aligned}$$

Set $u = -4x + 5$

Linear substitutions leading to building block Ia

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)} \\ &= -\frac{1}{4}\int u^{-1}\mathrm{d}u = -\frac{1}{4}\ln|u| + C \\ &= -\frac{1}{4}\ln|-4x+5| + C \quad .\end{aligned}$$

Set $u = -4x + 5$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)} \\ &= -\frac{1}{4}\int u^{-1}\mathrm{d}u = -\frac{1}{4}\ln|u| + C \\ &= -\frac{1}{4}\ln|-4x+5| + C \quad .\end{aligned}$$

Set $u = -4x + 5$

Lin. subst. leading to building block Ia: general case

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$\begin{aligned}
\int \frac{1}{ax+b}\mathrm{d}x &= \int \frac{1}{(\ \ ax+b)}\frac{\mathrm{d}(\ \ ax)}{a} \\
&= \int \frac{1}{(\ \ ax+b)}\frac{\mathrm{d}(\ \ ax+b)}{a} && \Big|\ \text{Set } u = ax+b \\
&= \int \frac{1}{u}\frac{\mathrm{d}u}{a} \\
&= \frac{1}{a}\int u^{-1}\mathrm{d}u = \frac{1}{a}\ln|u| + C \\
&= \frac{1}{a}\ln|ax+b| + C \quad .
\end{aligned}$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$, $n \neq 1$.

Example (Block Ib)

$$\int \frac{1}{x^n} \mathrm{d}x$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$, $n \neq 1$.

Example (Block Ib)

$$\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$, $n \neq 1$.

Example (Block Ib)

$$\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x =$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$, $n \neq 1$.

Example (Block Ib)

$$\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C$, $n \neq 1$.

Example

Integrate

$$\int \frac{1}{(3x+5)^3} \mathrm{d}x$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\int \frac{1}{(3x+5)^3}\mathrm{d}x = \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3}$$

Linear substitutions leading to building block Ib

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{(3x+5)^3}\mathrm{d}x &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3}\\ &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x+5)}{3}\end{aligned}$$

Linear substitutions leading to building block Ib

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{(3x+5)^3}\mathrm{d}x &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3} \\ &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x+5)}{3} \\ &= \int \frac{1}{u^3}\frac{\mathrm{d}u}{3}\end{aligned}$$

Set $u = 3x + 5$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{(3x+5)^3} dx &= \int \frac{1}{(3x+5)^3} \frac{d(3x)}{3} \\ &= \int \frac{1}{(3x+5)^3} \frac{d(3x+5)}{3} \\ &= \int \frac{1}{u^3} \frac{du}{3} \\ &= \frac{1}{3} \int u^{-3} du\end{aligned}$$

Set $u = 3x + 5$

Linear substitutions leading to building block Ib

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\begin{aligned}
\int \frac{1}{(3x+5)^3}\mathrm{d}x &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3} \\
&= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x+5)}{3} \qquad \Big| \text{ Set } u = 3x+5 \\
&= \int \frac{1}{u^3}\frac{\mathrm{d}u}{3} \\
&= \frac{1}{3}\int u^{-3}\mathrm{d}u = \frac{1}{3}\frac{u^{-2}}{(-2)} + C
\end{aligned}$$

Linear substitutions leading to building block Ib

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Integrate

$$\begin{aligned}
\int \frac{1}{(3x+5)^3}\mathrm{d}x &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3} \\
&= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x+5)}{3} \\
&= \int \frac{1}{u^3}\frac{\mathrm{d}u}{3} \\
&= \frac{1}{3}\int u^{-3}\mathrm{d}u = \frac{1}{3}\frac{u^{-2}}{(-2)} + C \\
&= -\frac{1}{6(3x+5)^2} + C \quad .
\end{aligned}$$

Set $u = 3x + 5$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

$$\begin{aligned}
\int \frac{1}{(3x+5)^3} \mathrm{d}x &= \int \frac{1}{(3x+5)^3} \frac{\mathrm{d}(3x)}{3} \\
&= \int \frac{1}{(3x+5)^3} \frac{\mathrm{d}(3x+5)}{3} \qquad \Big| \text{ Set } u = 3x+5 \\
&= \int \frac{1}{u^3} \frac{\mathrm{d}u}{3} \\
&= \frac{1}{3} \int u^{-3} \mathrm{d}u = \frac{1}{3} \frac{u^{-2}}{(-2)} + C \\
&= -\frac{1}{6(3x+5)^2} + C \quad .
\end{aligned}$$

Lin. subst. leading to building block Ib: general case

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Let $n \neq 1$. Integrate

$$\begin{aligned}
\int \frac{1}{(ax+b)^n} \mathrm{d}x &= \int \frac{1}{(ax+b)^n} \frac{\mathrm{d}(ax)}{a} \\
&= \int \frac{1}{(ax+b)^n} \frac{\mathrm{d}(ax+b)}{a} && \Big| \text{ Set } u = ax+b \\
&= \int \frac{1}{u^3} \frac{\mathrm{d}u}{a} \\
&= \frac{1}{a} \int u^{-n} \mathrm{d}u = -\frac{1}{a} \frac{u^{-n+1}}{(n-1)} + C \\
&= -\frac{1}{a(n-1)(ax+b)^{n-1}} + C \quad .
\end{aligned}$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\int \frac{x}{1+x^2}\mathrm{d}x$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\int \frac{x}{1+x^2}\mathrm{d}x = \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2}$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2}\\ &= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2}\end{aligned}$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2} \\ &= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2} \\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{2}\end{aligned}$$

Set $u = 1 + x^2$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx$. Building block IIIa: $\int \frac{1}{1+x^2} dx$.

Example (Block IIa)

$$\begin{aligned}
\int \frac{x}{1+x^2} dx &= \int \frac{1}{(1+x^2)} \frac{d(x^2)}{2} \\
&= \int \frac{1}{1+x^2} \frac{d(1+x^2)}{2} \\
&= \int \frac{1}{u} \frac{du}{2} \\
&= \frac{1}{2} \ln|u| + C
\end{aligned}$$

Set $u = 1 + x^2$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2}\\ &= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2} && \Big|\ \text{Set } u = 1+x^2\\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{2}\\ &= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C \quad .\end{aligned}$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}
\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2} \\
&= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2} \\
&= \int \frac{1}{u}\frac{\mathrm{d}u}{2} \\
&= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C \quad .
\end{aligned}$$

Set $u = 1 + x^2$

Example (Block IIIa)

$$\int \frac{1}{1+x^2}\mathrm{d}x$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2}\\ &= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2} && \Big| \text{ Set } u = 1+x^2\\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{2}\\ &= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C \quad .\end{aligned}$$

Example (Block IIIa)

$$\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

Example

$\int \frac{x}{2x^2+3}\mathrm{d}x$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
"Theoretical way" to solve example below: transform to IIa; this is slow.

Example

$\int \frac{x}{2x^2+3} \mathrm{d}x$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
"Theoretical way" to solve example below: transform to IIa; this is slow.

Feel free to skip slide, we will redo in next slide with a shortcut.

Example

$$\int \frac{x}{2x^2+3}\mathrm{d}x$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
"Theoretical way" to solve example below: transform to IIa; this is slow.

Feel free to skip slide, we will redo in next slide with a shortcut.

Example

$$
\begin{aligned}
\int \frac{x}{2x^2+3}\mathrm{d}x &= \int \frac{x}{3\left(\frac{2}{3}x^2+1\right)}\mathrm{d}x = \int \frac{x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)}\mathrm{d}x \\
&= \frac{3}{2}\int \frac{\sqrt{\frac{2}{3}}x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)}\mathrm{d}\left(\sqrt{\frac{2}{3}}x\right) \qquad \Bigg|\ \text{Set } u=\sqrt{\frac{2}{3}}x \\
&= \frac{1}{2}\int \frac{u}{u^2+1}\mathrm{d}u = \frac{1}{4}\ln(1+u^2)+C \\
&= \frac{1}{4}\ln\left(\frac{1}{3}(2x^2+3)\right)+C \\
&= \frac{1}{4}\ln(2x^2+3)+\frac{\ln\left(\frac{1}{3}\right)}{4}+C \\
&= \frac{1}{4}\ln(2x^2+3)+K \quad .
\end{aligned}
$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\int \frac{x}{2x^2+3} \mathrm{d}x$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1 + x^2) + C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\int \frac{x}{2x^2+3} dx = \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right)$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned}\int \frac{x}{2x^2+3}\mathrm{d}x &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{2x^2}{4}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned}\int \frac{x}{2x^2+3}\mathrm{d}x &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{2x^2+3}{4}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned}\int \frac{x}{2x^2+3}\mathrm{d}x &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{2x^2+3}{4}\right) \qquad \middle| \text{ Set } u = 2x^2+3\\ &= \frac{1}{4}\int \frac{1}{u}\mathrm{d}u\end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
The example below can be done directly, without transforming to block IIa.

Example

$$
\begin{aligned}
\int \frac{x}{2x^2+3} \mathrm{d}x &= \int \frac{1}{2x^2+3} \mathrm{d}\left(\frac{x^2}{2}\right) \\
&= \int \frac{1}{2x^2+3} \mathrm{d}\left(\frac{2x^2+3}{4}\right) \qquad \Big| \text{ Set } u = 2x^2+3 \\
&= \frac{1}{4} \int \frac{1}{u} \mathrm{d}u \\
&= \frac{1}{4} \ln|u| + C
\end{aligned}
$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned}
\int \frac{x}{2x^2+3} \mathrm{d}x &= \int \frac{1}{2x^2+3} \mathrm{d}\left(\frac{x^2}{2}\right) \\
&= \int \frac{1}{2x^2+3} \mathrm{d}\left(\frac{2x^2+3}{4}\right) \qquad \Big| \text{ Set } u = 2x^2+3 \\
&= \frac{1}{4} \int \frac{1}{u} \mathrm{d}u \\
&= \frac{1}{4} \ln|u| + C \\
&= \frac{1}{4} \ln(2x^2+3) + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\int \frac{1}{x^2+2}\mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\int \frac{1}{x^2+2}\mathrm{d}x = \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x\\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x\\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right)\\ &= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u\end{aligned}$$

Set $u = \frac{x}{\sqrt{2}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u\end{aligned}$$

Set $u = \frac{x}{\sqrt{2}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C.$

Example

$$
\begin{aligned}
\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{2}}\arctan(u) + C
\end{aligned}
$$

Set $u = \frac{x}{\sqrt{2}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C$.

Example

$$\begin{aligned}
\int \frac{1}{x^2+2} \mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} \mathrm{d}x \\
&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} \mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} \mathrm{d}u \\
&= \frac{1}{\sqrt{2}} \arctan(u) + C \\
&= \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C
\end{aligned}$$

Set $u = \frac{x}{\sqrt{2}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$. **Let $a > 0$.**

Example

$$\begin{aligned}
\int \frac{1}{x^2+a}\mathrm{d}x &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)}\sqrt{a}\,\mathrm{d}\left(\frac{x}{\sqrt{a}}\right) \qquad \Bigg|\ \text{Set } u = \frac{x}{\sqrt{a}} \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{a}}\arctan(u) + C \\
&= \frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right) + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1 + x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

  (where r is some number to be determined).
- To find p, q, we complete the square.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

  (where r is some number to be determined).
- To find p, q, we complete the square.
- In this way, integrals of the form $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).
- To find p, q, we complete the square.
- In this way, integrals of the form $\displaystyle\int \frac{Ax + B}{ax^2 + bx + c} \mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

$$\int \frac{x}{x^2 + x + 1} \mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square.

$$\int \frac{x}{x^2+x+1} \mathrm{d}x = \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square.

$$\int \frac{x}{x^2 + x + 1}\mathrm{d}x = \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\mathrm{d}x \\ &= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}\left(x+\frac{1}{2}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots ⇒ complete the square.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\mathrm{d}x \\ &= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}\left(x+\frac{1}{2}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\mathrm{d}x \\ &= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}\left(x+\frac{1}{2}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}
\int \frac{x}{x^2+x+1} \mathrm{d}x &= \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \mathrm{d}x \\
&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \mathrm{d}\left(x + \frac{1}{2}\right) \\
&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} \mathrm{d}u
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\mathrm{d}x \\
&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\mathrm{d}\left(x + \frac{1}{2}\right) \\
&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}}\mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C.$
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C.$

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}
\int \frac{x}{x^2+x+1} \mathrm{d}x &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2} x + \frac{1}{4} - \frac{1}{4} + 1} \mathrm{d}x \\
&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \mathrm{d}\left(x + \frac{1}{2}\right) \\
&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} \mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}} \mathrm{d}u - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} \mathrm{d}u
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\mathrm{d}x \\ &= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\mathrm{d}\left(x+\frac{1}{2}\right) \\ &= \int \frac{u-\frac{1}{2}}{u^2+\frac{3}{4}}\mathrm{d}u \\ &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\int \frac{x}{x^2+x+1} dx \quad = \quad \int \frac{u}{u^2+\frac{3}{4}} du - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}} du$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

$$\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

$$\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u = \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1 + x^2) + C.$
Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\int \frac{x}{x^2 + x + 1} dx = \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du$$

$$\int \frac{1}{u^2 + \frac{3}{4}} du = \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2 + 1\right)} du$$

$$= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2 + 1\right)} \frac{\sqrt{3}}{2} d\left(\frac{2u}{\sqrt{3}}\right)$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

$$\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u = \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u$$

$$= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2u}{\sqrt{3}}\right)$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u \\
&= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)} \frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2u}{\sqrt{3}}\right) \\
&= \frac{2\sqrt{3}}{3}\int \frac{1}{z^2+1}\mathrm{d}z
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u \\
&= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2u}{\sqrt{3}}\right) \\
&= \frac{2\sqrt{3}}{3}\int \frac{1}{z^2+1}\mathrm{d}z = \frac{2\sqrt{3}}{3}\arctan z + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1} dx &= \int \frac{u}{u^2+\frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2+\frac{3}{4}} du \\
\int \frac{1}{u^2+\frac{3}{4}} du &= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)} du \\
&= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)} \frac{\sqrt{3}}{2} d\left(\frac{2u}{\sqrt{3}}\right) \\
&= \frac{2\sqrt{3}}{3} \int \frac{1}{z^2+1} dz = \frac{2\sqrt{3}}{3} \arctan z + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned} \int \frac{x}{x^2+x+1} \mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}} \mathrm{d}u - \frac{1}{2} \int \frac{1}{u^2+\frac{3}{4}} \mathrm{d}u \\ &= \int \frac{u}{u^2+\frac{3}{4}} \mathrm{d}u - \frac{1}{2} \frac{2\sqrt{3}}{3} \arctan z + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots ⇒ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\ &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C\end{aligned}$$

$$\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned} \int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\ &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\ \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}dx = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}dx &= \int \frac{u}{u^2+\frac{3}{4}}du - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}du \\
&= \int \frac{u}{u^2+\frac{3}{4}}du - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
\int \frac{u}{u^2+\frac{3}{4}}du &= \int \frac{1}{u^2+\frac{3}{4}}d\left(\frac{u^2}{2}\right) \\
&= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}d\left(u^2 \qquad \right)
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right) \\
&= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(u^2 {\color{red}+\frac{3}{4}}\right)
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C.$
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C.$

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right) \\
&= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(u^2+\frac{3}{4}\right) = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
&= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right) \\
&= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(u^2+\frac{3}{4}\right) = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned} \int \frac{x}{x^2+x+1} \mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}} \mathrm{d}u - \frac{1}{2} \int \frac{1}{u^2+\frac{3}{4}} \mathrm{d}u \\ &= \frac{1}{2} \ln\left(u^2 + \frac{3}{4}\right) - \frac{1}{2} \frac{2\sqrt{3}}{3} \arctan z + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x+\frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u\\ &= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C\\ &= \frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}\int \frac{x}{x^2+x+1} dx &= \int \frac{u}{u^2+\frac{3}{4}} du - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}} du \\ &= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\ &= \frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x+\frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u \\
&= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
&= \frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C \\
&= \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)+C$.

Building block IIIa: $\int \frac{1}{1+x^2}dx = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}dx &= \int \frac{u}{u^2+\frac{3}{4}}du - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}du \\
&= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C \\
&= \frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C \\
&= \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\int \frac{x}{(x^2+1)^n} \mathrm{d}x$$

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\int \frac{x}{(x^2+1)^n}\mathrm{d}x = \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2}$$

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\begin{aligned}\int \frac{x}{(x^2+1)^n}\mathrm{d}x &= \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2}\\ &= \frac{1}{2}\int u^{-n}\mathrm{d}u\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\begin{aligned}
\int \frac{x}{(x^2+1)^n}\mathrm{d}x &= \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2} \\
&= \frac{1}{2}\int u^{-n}\mathrm{d}u \\
&= \begin{cases} & \text{if } n = 1 \\ & \text{if } n \neq 1 \end{cases},
\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\begin{aligned}\int \frac{x}{(x^2+1)^n}\mathrm{d}x &= \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2}\\ &= \frac{1}{2}\int u^{-n}\mathrm{d}u\\ &= \begin{cases} \frac{1}{2}\ln(x^2+1)+C & \text{if } n=1\\ & \text{if } n\neq 1\end{cases},\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\begin{aligned}\int \frac{x}{(x^2+1)^n}\mathrm{d}x &= \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2}\\ &= \frac{1}{2}\int u^{-n}\mathrm{d}u\\ &= \begin{cases} \frac{1}{2}\ln(x^2+1)+C & \text{if } n=1\\ & \text{if } n\neq 1\end{cases},\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\begin{aligned}\int \frac{x}{(x^2+1)^n}\mathrm{d}x &= \int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2}\\ &= \frac{1}{2}\int u^{-n}\mathrm{d}u\\ &= \begin{cases} \frac{1}{2}\ln(x^2+1)+C & \text{if } n=1\\ \frac{1}{2}\frac{(x^2+1)^{-n+1}}{(-n+1)}+C & \text{if } n\neq 1\end{cases},\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \int \frac{1}{x^2+1}\mathrm{d}x$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned} \arctan x + C &= \int \frac{1}{x^2+1} dx \\ &= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\ &= \frac{1}{x^2+1}x - \int x\,\mathrm{d}\left(\frac{1}{x^2+1}\right)\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$
\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\
&= \frac{1}{x^2+1}x - \int x\mathrm{d}\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\mathbf{?}
\end{aligned}
$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1} dx \\
&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x \left(-\frac{2x}{(x^2+1)^2}\right) dx
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\
&= \frac{1}{x^2+1}x - \int x\mathrm{d}\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\left(-\frac{2x}{(x^2+1)^2}\right)\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{x^2}{(x^2+1)^2}\mathrm{d}x
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1} dx \\
&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x \left(-\frac{2x}{(x^2+1)^2}\right) dx \\
&= \frac{x}{x^2+1} + 2\int \frac{-1 + x^2 + 1}{(x^2+1)^2} dx
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1} dx \\
&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x \left(-\frac{2x}{(x^2+1)^2}\right) dx \\
&= \frac{x}{x^2+1} + 2\int \frac{-1 + x^2+1}{(x^2+1)^2} dx \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1} dx - 2\int \frac{1}{(x^2+1)^2} dx
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$
\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1} dx \\
&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x \left(-\frac{2x}{(x^2+1)^2}\right) dx \\
&= \frac{x}{x^2+1} + 2\int \frac{-1+x^2+1}{(x^2+1)^2} dx \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1} dx - 2\int \frac{1}{(x^2+1)^2} dx
\end{aligned}
$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\
&= \frac{1}{x^2+1}x - \int x \mathrm{d}\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\left(-\frac{2x}{(x^2+1)^2}\right)\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{-1+x^2+1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\mathrm{d}x - 2\int \frac{1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\
&= \frac{1}{x^2+1}x - \int x\mathrm{d}\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\left(-\frac{2x}{(x^2+1)^2}\right)\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{-1+x^2+1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\mathrm{d}x - 2\int \frac{1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{dx}{(x^2+1)^2}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{dx}{(x^2+1)^2}$$

Rearrange terms

$$2\int \frac{dx}{(1+x^2)^2} = \left(\frac{x}{x^2+1} + \arctan x\right) + C' \quad .$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}$$

Rearrange terms

$$2\int \frac{\mathrm{d}x}{(1+x^2)^2} = \left(\frac{x}{x^2+1} + \arctan x\right) + C' \quad .$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C \quad = \quad \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}$$

Rearrange terms and divide by 2 to get the desired integral:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan x\right) + C'' \quad .$$

Building block IIIb

- Building block IIIa:

$$\int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

Building block IIIb

- Building block IIIa:

$$\int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$\int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

Building block IIIb

- Building block IIIa:

$$\int \frac{1}{(x^2+1)} dx = \arctan x + C \quad .$$

- Block IIIb:

$$\int \frac{1}{(x^2+1)^n} dx$$

- Unlike other cases, IIIb is much harder than IIIa.

Building block IIIb

- Building block IIIa:

$$\int \frac{1}{(x^2+1)}\mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$.

Building block IIIb

- Building block IIIa:

$$\int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We are looking for a formula for $J(n)$.

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).
- We start by $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x$ and integrate by parts.

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).
- We start by $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x$ and integrate by parts.
- In this way we end up expressing $J(n)$ via $J(n-1)$.

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)}\mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).
- We start by $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}}\mathrm{d}x$ and integrate by parts.
- In this way we end up expressing $J(n)$ via $J(n-1)$.
- We work our way from $J(n)$ to $J(n-1)$, from $J(n-1)$ to $J(n-2)$, and so on, until we get to $J(1)$.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right)
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}}\right)
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$\begin{aligned} J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\ &= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\ &= \frac{x}{(x^2+1)^{n-1}} - \int x \mathbf{?} \end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$\begin{aligned} J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\ &= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\ &= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$\begin{aligned} J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\ &= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\ &= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\ &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{x^2}{(1+x^2)^n} \mathrm{d}x \end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}} \right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}} \right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
&\quad - 2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}}\right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2 - 1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
&\quad - 2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}} \right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
&\quad - 2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) J(n-1) - 2(n-1) J(n) \quad .
\end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$. We have that:

$$\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}}\mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}}x - \int x\mathrm{d}\left(\frac{1}{(1+x^2)^{n-1}}\right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x\frac{(-n+1)2x}{(1+x^2)^n}\mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{1+x^2-1}{(1+x^2)^n}\mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{1}{(1+x^2)^{n-1}}\mathrm{d}x \\
&\quad - 2(n-1)\int \frac{1}{(1+x^2)^n}\mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .
\end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \frac{(-n+1)2x}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
&\quad - 2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .
\end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) \quad = \quad \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$2(n-1)J(n) = \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1)$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) \quad = \quad \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$2(n-1)J(n) \quad = \quad \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1)$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$.

Example

Recall that $J(n)=\int \frac{1}{(x^2+1)^n}\mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$J(n)=\frac{x}{(2n-2)(x^2+1)^{n-1}}+\frac{2n-3}{2n-2}J(n-1)$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$2(n-1)J(n) = \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1)$$

$$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad .$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2) \right)$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2)\right) = \dots$

and so on.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2)\right) = \dots$

and so on. A formula for the final result can be written using the above (found in Calculus for beginners, Chapter "Techniques of integration").

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} \mathrm{d}x$	$\int \frac{1}{x^n} \mathrm{d}x$	$\int \frac{A}{ax+b} \mathrm{d}x$	$\int \frac{A}{(ax+b)^n} \mathrm{d}x$
II	$\int \frac{x}{x^2+1} \mathrm{d}x$	$\int \frac{x}{\left(x^2+1\right)^n} \mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} \mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} \mathrm{d}x$
III	$\int \frac{1}{x^2+1} \mathrm{d}x$	$\int \frac{1}{\left(x^2+1\right)^n} \mathrm{d}x$	$\int \frac{B}{ax^2+bx+c} \mathrm{d}x$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} \mathrm{d}x$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{\left(x^2+1\right)^n} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{\left(x^2+1\right)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x}\mathrm{d}x$	$\int \frac{1}{x^n}\mathrm{d}x$	$\int \frac{A}{ax+b}\mathrm{d}x$	$\int \frac{A}{(ax+b)^n}\mathrm{d}x$
II	$\int \frac{x}{x^2+1}\mathrm{d}x$	$\int \frac{x}{\left(x^2+1\right)^n}\mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c}\mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n}\mathrm{d}x$
III	$\int \frac{1}{x^2+1}\mathrm{d}x$	$\int \frac{1}{\left(x^2+1\right)^n}\mathrm{d}x$	$\int \frac{B}{ax^2+bx+c}\mathrm{d}x$	$\int \frac{B}{\left(ax^2+bx+c\right)^n}\mathrm{d}x$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{\left(x^2+1\right)^n} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{\left(x^2+1\right)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{\left(x^2+1\right)^n} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{\left(x^2+1\right)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
    - Block I, linear substitutions: done in full detail.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{\left(x^2+1\right)^n} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{\left(x^2+1\right)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x}dx$	$\int \frac{1}{x^n}dx$	$\int \frac{A}{ax+b}dx$	$\int \frac{A}{(ax+b)^n}dx$
II	$\int \frac{x}{x^2+1}dx$	$\int \frac{x}{\left(x^2+1\right)^n}dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c}dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n}dx$
III	$\int \frac{1}{x^2+1}dx$	$\int \frac{1}{\left(x^2+1\right)^n}dx$	$\int \frac{B}{ax^2+bx+c}dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n}dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail. To derive substitution: complete the square.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{\left(x^2+1\right)^n} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{\left(x^2+1\right)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail. To derive substitution: complete the square.
  - Block IIb, IIIb, linear substitutions: to derive substitution: again, complete the square;

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x}dx$	$\int \frac{1}{x^n}dx$	$\int \frac{A}{ax+b}dx$	$\int \frac{A}{(ax+b)^n}dx$
II	$\int \frac{x}{x^2+1}dx$	$\int \frac{x}{\left(x^2+1\right)^n}dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c}dx$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n}dx$
III	$\int \frac{1}{x^2+1}dx$	$\int \frac{1}{\left(x^2+1\right)^n}dx$	$\int \frac{B}{ax^2+bx+c}dx$	$\int \frac{B}{\left(ax^2+bx+c\right)^n}dx$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail. To derive substitution: complete the square.
  - Block IIb, IIIb, linear substitutions: to derive substitution: again, complete the square; computations are analogous and we leave them for exercise.