Freecalc
Homework on Lecture 6
Will be quizzed: time to be announced in class

1. Integrate.

   (a) $\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x.$

   answer: $\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}\right)\right)+C$

   (b) $\displaystyle\int \frac{1}{3+\sin x}\mathrm{d}x.$

   answer: $\frac{1}{\sqrt{2}}\arctan\left(\frac{3\tan\left(\frac{x}{2}\right)+1}{2\sqrt{2}}\right)$

   (c) $\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x.$ (Hint: this integral can be done simply with the substitution $x=\arctan t$.)

   answer: $\frac{1}{5}\ln(\sin x+2\cos x)+\frac{2}{5}x+C$

Solution. 1.a This integral is of none of the forms that can be integrated quickly. Therefore we have to use the standard rationalizing substitution $x=2\arctan t$, $t=\tan\left(\frac{x}{2}\right)$. We recall that from the double angle formulas it follows that

$$\cos(2\arctan t)=\frac{\cos^2(\arctan t)-\sin^2(2\arctan t)}{\cos^2(\arctan t)+\sin^2(\arctan t)}=\frac{1-t^2}{1+t^2}\quad .$$

Therefore we can solve the integral as follows.

$$\begin{array}{rcll}\displaystyle\int\frac{1}{3+\cos x}\mathrm{d}x &=& \displaystyle\int\frac{1}{3+\cos(2\arctan t)}\mathrm{d}\left(2\arctan t\right) & \Big|\text{ Substitute } x=2\arctan t\\ &=& \displaystyle\int\frac{1}{\left(3+\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}\mathrm{d}t & \\ &=& \displaystyle\int\frac{2}{4+2t^2}\mathrm{d}t & \\ &=& \displaystyle\int\frac{1}{2+t^2}\mathrm{d}t & \\ &=& \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}t\right)+C & \\ &=& \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\tan\left(\frac{x}{2}\right)\right)+C\quad . & \end{array}$$

Solution. 1.c This integral is of none of the forms that can be integrated quickly. Therefore we can solve it using the standard rationalizing substitution $x=2\arctan t$, $t=\tan\left(\frac{x}{2}\right)$. This results in somewhat long computations and we invite the reader to try it.

However, as proposed in the hint, the substitution $x = \arctan t$ works much faster:

$$\begin{array}{rcll}
\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x &=& \displaystyle\int \frac{1}{2+\tan(\arctan t)}\mathrm{d}\left(\arctan t\right) & \Big| \text{ Substitute } x = \arctan t \\
&=& \displaystyle\int \frac{1}{(2+t)}\frac{1}{(1+t^2)}\mathrm{d}t & \Big| \text{ Decompose into partial fractions} \\
&=& \displaystyle\int \left(\frac{\frac{1}{5}}{(t+2)} + \frac{-\frac{t}{5}+\frac{2}{5}}{(t^2+1)} \right) \mathrm{d}t & \\
&=& \displaystyle\frac{1}{5}\ln(t+2) - \frac{1}{10}\ln(t^2+1) + \frac{2}{5}\arctan t + C & \Big| \text{ Substitute back } t = \tan x \\
&=& \displaystyle\frac{1}{5}\ln(\tan x+2) - \frac{1}{10}\ln(\tan^2 x+1) + \frac{2}{5}x + C & \\
&=& \displaystyle\frac{1}{5}\ln(\tan x+2) + \frac{1}{5}\ln(\cos x) + \frac{2}{5}x + C & \\
&=& \displaystyle\frac{1}{5}\ln\left((\tan x+2)\cos x\right) + \frac{2}{5}x + C & \\
&=& \displaystyle\frac{1}{5}\ln\left(\sin x+2\cos x\right) + \frac{2}{5}x + C. &
\end{array}$$

2. Integrate. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\int \sin(3x)\cos(2x)\mathrm{d}x$.

   answer: $-\frac{1}{10}\cos(5x) - \frac{1}{2}\cos x + C$

   (b) $\displaystyle\int \sin x\cos(5x)\mathrm{d}x$.

   answer: $-\frac{1}{12}\cos(6x) + \frac{1}{8}\cos(4x) + C$

   (c) $\displaystyle\int \cos(3x)\sin(2x)\mathrm{d}x$.

   answer: $-\frac{1}{10}\cos(5x) + \frac{1}{2}\cos x + C$

   (d) $\displaystyle\int \sin(5x)\sin(3x)\mathrm{d}x$.

   answer: $\frac{1}{4}\sin(2x) - \frac{1}{16}\sin(8x) + C$

   (e) $\displaystyle\int \cos(x)\cos(3x)\mathrm{d}x$.

   answer: $\frac{1}{8}\sin(4x) + \frac{1}{4}\sin(2x) + C$

3. Integrate.

   (a) $\displaystyle\int \sin^2 x\cos x\mathrm{d}x$.

   answer: $\frac{1}{3}\sin^3 x + C$

   (b) $\displaystyle\int \sin^2 x\mathrm{d}x$.

   answer: $\frac{x}{2} - \frac{1}{4}\sin(2x) + C$

   (c) $\displaystyle\int \cos^3 x\mathrm{d}x$.

   answer: $\sin x - \frac{1}{3}\sin^3 x + C$

4. Integrate

   (a) $\displaystyle\int \sec^3 x\mathrm{d}x$.

   answer: $\frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C$

   (b) $\displaystyle\int \tan^3 x\mathrm{d}x$.

   answer: $\frac{1}{2}\tan^2 x - \ln|\sec x| + C$

   (c) $\displaystyle\int \sec^2 x\tan^2 x\mathrm{d}x$.

   answer: $\frac{\tan^3 x}{3} + C$