Math 141

Lecture 6

Greg Maloney

Todor Milev

University of Massachusetts Boston

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Outline

1. Trigonometric Integrals
   - Integrating rational trigonometric integrals
   - Ad hoc methods for trigonometric integrals

Integrals of the form $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$*?*

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- Yes. We will learn how in what follows.

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  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.

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Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.
  - Solve as previously studied.

The rationalizing substitution $\theta = 2\arctan t$

Recall the expression of $\sin(2z), \cos(2z)$ via $\tan z$:

$$\sin(2z) =$$

$$\cos(2z)$$

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$$\sin(2z) = 2\sin z\cos z = \frac{2\sin z\cos z}{(\cos^2 z + \sin^2 z)}$$

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Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta) d\theta$ can be integrated via the substitution $\theta = 2\arctan t$.

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Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta)d\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$?

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Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta) d\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$? How does this transform $d\theta$?

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$d\theta$

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t &=
\end{aligned}$$

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t &= \tan\left(\frac{\theta}{2}\right)
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\mathrm{d}\theta &= 2\mathrm{d}(\arctan t) = \frac{2}{1+t^2}\,\mathrm{d}t\\
t &= \tan\left(\frac{\theta}{2}\right)
\end{aligned}$$

Theorem

The substitution given above transforms $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$ *to an integral of a rational function of* t.

Example

$$\int \frac{d\theta}{2\sin\theta - \cos\theta + 5}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{d\theta}{2\sin\theta - \cos\theta + 5} = \int \frac{2dt}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}$$

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$$= \int \frac{2dt}{6t^2 + 4t + 4}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}\\ &= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4}\\ &= \int \frac{\mathrm{d}t}{3t^2 + 2t + 2}\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{d\theta}{2\sin\theta - \cos\theta + 5} = \int \frac{2dt}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}$$

$$= \int \frac{2dt}{6t^2 + 4t + 4}$$

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$$= \int \frac{dt}{3\left(t^2 + 2t\frac{1}{3} \qquad + \frac{2}{3}\right)}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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&= \int \frac{2dt}{6t^2+4t+4} \\
&= \int \frac{dt}{3t^2+2t+2} \\
\text{(complete square)} \quad &= \int \frac{dt}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)}
\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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&= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4} \\
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&= \int \frac{\mathrm{d}t}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t + \frac{1}{3}\right)^2 + \frac{5}{9}}
\end{aligned}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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&= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2 + \frac{5}{9}} \\
&= \frac{1}{3}\int \frac{dt}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)}
\end{aligned}
$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
&= \int \frac{2\mathrm{d}t}{6t^2+4t+4} \\
&= \int \frac{\mathrm{d}t}{3t^2+2t+2} \\
&= \int \frac{\mathrm{d}t}{3\left(t^2+2t\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right)} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}
\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{d\theta}{2\sin\theta - \cos\theta + 5} = \frac{1}{3}\int \frac{dt}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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$$= \frac{3}{5}\int \frac{d\left(\quad (t \quad)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2 + 1\right)}$$

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&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t \qquad \right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2 + 1\right)}
\end{aligned}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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$$= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}d\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t + \frac{1}{3}\right)$.

$$\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)} \\
&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2 + 1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1}
\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

$$\begin{aligned}
\int \frac{d\theta}{2\sin\theta - \cos\theta + 5} &= \frac{1}{3}\int \frac{dt}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)} \\
&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3} d\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{dz}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C
\end{aligned}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

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&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3} d\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{dz}{z^2+1} \\
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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

$$\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)} \\
&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right) + C
\end{aligned}$$

The integral $\int \sec\theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $d\theta = 2\frac{1}{1+t^2}dt$.

$$\int \sec\theta d\theta \quad =$$

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$$\int \sec\theta \mathrm{d}\theta = \int \frac{1}{\cos\theta}\mathrm{d}\theta$$

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$$\int \sec\theta \mathrm{d}\theta = \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)}\mathrm{d}t$$

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$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt$$

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$$\begin{aligned}\int \sec\theta d\theta &= \int \frac{1}{\cos\theta}d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}dt\\ &= \int \frac{2}{1-t^2}dt\end{aligned}$$

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$$\begin{aligned}\int \sec\theta d\theta &= \int \frac{1}{\cos\theta}d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}dt \\ &= \int \frac{2}{1-t^2}dt = \int \left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt \qquad \text{part. fractions}\end{aligned}$$

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$$\begin{aligned}
\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt \\
&= \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \qquad \text{part. fractions} \\
&= -\ln|1-t| + \ln|1+t| + C
\end{aligned}$$

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\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt \\
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$$
\begin{aligned}
\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt \\
&= \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt \qquad \text{part. fractions} \\
&= -\ln|1-t| + \ln|1+t| + C \\
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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

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$$\int \sec\theta d\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta =$$

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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

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$$\tan\theta + \sec\theta = \frac{\sin\theta + 1}{\cos\theta}$$

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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

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$$\tan\theta + \sec\theta = \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

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$$\int \sec\theta d\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta = \frac{\sin\theta + {\color{red}1}}{\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + {\color{red}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

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Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $d\theta = 2\dfrac{1}{1+t^2}dt$.

$$\int \sec\theta d\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta = \frac{\sin\theta + 1}{\color{red}{\cos\theta}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\color{red}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}}$$

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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\begin{aligned}\tan\theta+\sec\theta &= \frac{\sin\theta+1}{\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} \\ &= \frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}\end{aligned}$$

The integral $\int \sec\theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $d\theta = 2\frac{1}{1+t^2}dt$.

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Trigonometric Integrals - quick ad hoc techniques

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- Particular integral types may be computable with quicker ad hoc techniques.
- We illustrate such techniques on examples.
- Examples to which our ad hoc techniques apply arise from integrals needed outside of the subject of Calculus II, so these techniques are important.
- The trigonometric integral we saw, $\int \frac{d\theta}{2\sin\theta - \cos\theta + 5}$, will not work with any of following ad-hoc techniques, so the general method is important as well.

Example

$$\int \sin^3 x dx$$

Example

$$\int \sin^3 x \mathrm{d}x = \int \sin^2 x \sin x \mathrm{d}x$$

Example

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$
$$= \int \sin^2 x d(\textbf{?} \qquad)$$

Example

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx \\ &= \int \sin^2 x\,d(-\cos x)\end{aligned}$$

Example

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\ &= \int \sin^2 x d(-\cos x) \\ &= \int (-1)\left(\textbf{?} \qquad\qquad \right) d(\cos x)\end{aligned}$$

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Can we rewrite $\sin^2 x$ via $\cos x$?

Example

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

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Can we rewrite $\sin^2 x$ via $\cos x$?

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$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\ &= \int \sin^2 x d(-\cos x) \\ &= \int (-1)\left(1-\cos^2 x\right) d(\cos x) \\ &= \int \left(\cos^2 x - 1\right) d(\cos x)\end{aligned}$$

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$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

$$= \int \sin^2 x d(-\cos x) \qquad \text{Can we rewrite } \sin^2 x \text{ via } \cos x?$$

$$= \int (-1)\left(1 - \cos^2 x\right) d(\cos x)$$

$$= \int \left(\cos^2 x - 1\right) d(\cos x) \qquad \text{Set } u = \cos x$$

$$= \int \left(u^2 - 1\right) du$$

Example

$$
\begin{aligned}
\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
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&= \int \left(\cos^2 x - 1\right) d(\cos x) \\
&= \int \left(u^2 - 1\right) du \\
&= \frac{u^3}{3} - u + C
\end{aligned}
$$

Can we rewrite $\sin^2 x$ via $\cos x$?

Set $u = \cos x$

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\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
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&= \frac{1}{3}\cos^3 x - \cos x + C \quad .
\end{aligned}$$

Can we rewrite $\sin^2 x$ via $\cos x$?

Set $u = \cos x$

Example

$$\int \cos^5 x \sin^2 x dx$$

Example

$$\int \cos^5 x \sin^2 x \mathrm{d}x = \int \cos^4 x \sin^2 x \cos x \mathrm{d}x$$

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$$\int \cos^5 x \sin^2 x dx = \int \cos^4 x \sin^2 x \cos x dx$$
$$= \int \cos^4 x \sin^2 x d(\textbf{?} \quad)$$

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x\\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Example

$$\int \cos^5 x \sin^2 x dx = \int \cos^4 x \sin^2 x \cos x dx$$
$$= \int \cos^4 x \sin^2 x d(\sin x)$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx\\ &= \int \cos^4 x \sin^2 x d(\sin x)\\ &= \int \left(\cos^2 x\right)^2 \sin^2 x d(\sin x)\end{aligned}$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\ &= \int \cos^4 x \sin^2 x d(\sin x) \\ &= \int \left(\cos^2 x\right)^2 \sin^2 x d(\sin x) \\ &= \int \left(1 - \sin^2 x\right)^2 \sin^2 x d(\sin x)\end{aligned}$$

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&= \int \left(1 - u^2\right)^2 u^2 du
\end{aligned}$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\ &= \int \cos^4 x \sin^2 x d(\sin x) \\ &= \int \left(\cos^2 x\right)^2 \sin^2 x d(\sin x) \\ &= \int \left(1 - \sin^2 x\right)^2 \sin^2 x d(\sin x) \\ &= \int \left(1 - u^2\right)^2 u^2 du \\ &= \int \left(1 - 2u^2 + u^4\right) u^2 du\end{aligned}$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x\\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x)\\ &= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)\\ &= \int \left(1 - \sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)\\ &= \int \left(1-u^2\right)^2 u^2 \mathrm{d}u\\ &= \int \left(1 - 2u^2 + u^4\right) u^2 \mathrm{d}u\\ &= \int \left(u^2 - 2u^4 + u^6\right) \mathrm{d}u\end{aligned}$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\ &= \int \cos^4 x \sin^2 x d(\sin x) \\ &= \int \left(\cos^2 x\right)^2 \sin^2 x d(\sin x) \\ &= \int \left(1 - \sin^2 x\right)^2 \sin^2 x d(\sin x) \\ &= \int \left(1 - u^2\right)^2 u^2 du \\ &= \int \left(1 - 2u^2 + u^4\right) u^2 du \\ &= \int \left(u^2 - 2u^4 + u^6\right) du \\ &= \textbf{?}\end{aligned}$$

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\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
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&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C
\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

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\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\
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&= \int \left(u^2-2u^4+u^6\right) du \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C \quad .
\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Set $u = \sin x$

$$\int \sin^m x \cos^n x \mathrm{d}x$$

When n – odd:

$$\int \sin^m x \cos^n x \mathrm{d}x$$

When m – odd:

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$$

When n – odd:
$\cos x dx$
$= d(\sin x)$

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When n – odd:
$\cos x dx$
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Express $\cos x$
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$$= \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$$

When n – odd: $\cos x dx = d(\sin x)$

Express $\cos x$ via $\sin x$

Set $\sin x = u$

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When n – odd: $\cos x dx = d(\sin x)$

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$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$$

When m – odd: $\sin x dx = d(-\cos x)$

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x) = \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x) = \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$$

When n – odd: $\cos x dx = d(\sin x)$

Express $\cos x$ via $\sin x$

Set $\sin x = u$

$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x) = -\int \left(1 - \cos^2 x\right)^{\frac{m-1}{2}} \cos^n x d(\cos x)$$

When m – odd: $\sin x dx = d(-\cos x)$

Express $\cos x$ via $\sin x$

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$$
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When n – odd: $\cos x dx = d(\sin x)$. Express $\cos x$ via $\sin x$. Set $\sin x = u$

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When m – odd: $\sin x dx = d(-\cos x)$. Express $\cos x$ via $\sin x$

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$$	When n – odd: $\cos x dx = d(\sin x)$
$$= \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$$	Express $\cos x$ via $\sin x$
$$= \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$$	Set $\sin x = u$
$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$$	When m – odd: $\sin x dx = d(-\cos x)$
$$= -\int \left(1 - \cos^2 x\right)^{\frac{m-1}{2}} \cos^n x d(\cos x)$$	Express $\cos x$ via $\sin x$

$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$	When n – odd: $\cos x dx = d(\sin x)$
$= \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$	Express $\cos x$ via $\sin x$
$= \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$	Set $\sin x = u$
$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$	When m – odd: $\sin x dx = d(-\cos x)$
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$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x) = \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x) = \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$$

When n – odd: $\cos x dx = d(\sin x)$. Express $\cos x$ via $\sin x$. Set $\sin x = u$

$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x) = -\int \left(1 - \cos^2 x\right)^{\frac{m-1}{2}} \cos^n x d(\cos x) = -\int \left(1 - u^2\right)^{\frac{m-1}{2}} u^n du$$

When m – odd: $\sin x dx = d(-\cos x)$. Express $\cos x$ via $\sin x$. Set $\cos x = u$

If both m, n- even,

$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$	When n – odd: $\cos x dx = d(\sin x)$
$= \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$	Express $\cos x$ via $\sin x$
$= \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$	Set $\sin x = u$
$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$	When m – odd: $\sin x dx = d(-\cos x)$
$= -\int \left(1 - \cos^2 x\right)^{\frac{m-1}{2}} \cos^n x d(\cos x)$	Express $\cos x$ via $\sin x$
$= -\int \left(1 - u^2\right)^{\frac{m-1}{2}} u^n du$	Set $\cos x = u$

If both m, n- even, use $\begin{array}{|lcl} \sin^2 x &=& \frac{1-\cos(2x)}{2} \\ \cos^2 x &=& \frac{\cos(2x)+1}{2} \end{array}$ and substitute $s = 2x$ to lower trig powers. Repeat above considerations.

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$$

$$= \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$$

$$= \int u^m \left(1 - u^2\right)^{\frac{n-1}{2}} du$$

When n – odd: $\cos x dx = d(\sin x)$

Express $\cos x$ via $\sin x$

Set $\sin x = u$

$$\int \sin^m x \cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$$

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$$\begin{aligned} \sin^2 x &= \frac{1-\cos(2x)}{2} \\ \cos^2 x &= \frac{\cos(2x)+1}{2} \end{aligned}$$

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Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx$$

Example

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) \mathrm{d}x$$

express $\sin^2 x$ via $\cos(2x)$

Example

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$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) \mathrm{d}x$$

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_0^{\frac{\pi}{2}}$$

express $\sin^2 x$ via $\cos(2x)$

Example

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)\mathrm{d}x$$

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Example

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) dx$$

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Example

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right) dx$$

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Example

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx$$

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Example

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)\mathrm{d}x \\ &= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \mathbf{?}\ .\end{aligned}$$

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Example

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) \mathrm{d}x$$

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express $\sin^2 x$ via $\cos(2x)$

Example

$$\int_{t=0}^{t=1} \sqrt{1-t^2}\mathrm{d}t$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) dx$$

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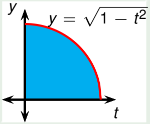
Example



$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx$$

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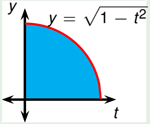
Example

Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$.



$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) \mathrm{d}x$$

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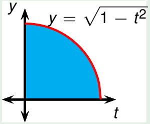


$$\int_{t=0}^{t=1} \sqrt{1 - t^2} \mathrm{d}t$$

Example

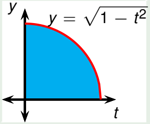
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx \\ &= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{4} - \frac{\sin \pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \frac{\pi}{4}.\end{aligned}$$

express $\sin^2 x$ via $\cos(2x)$

Example

Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$. Then

$dt = d(\cos x) = -\sin x dx$.



$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx$$

| express $\sin^2 x$ via $\cos(2x)$

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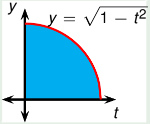

$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1 - \cos^2 x} \sin x dx$$

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) dx$$

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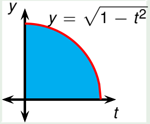

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Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right) dx$$

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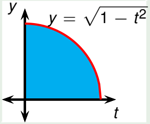

$$\int_{t=0}^{t=1} \sqrt{1-t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x dx$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx$$

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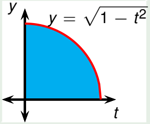

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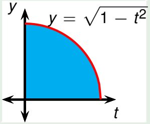

$$\int_{t=0}^{t=1} \sqrt{1-t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x dx$$

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Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2}\right) dx \quad \Big| \quad \text{express } \sin^2 x \text{ via } \cos(2x)$$

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{2}}$$

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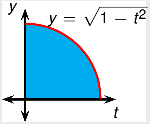

$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1 - \cos^2 x} \sin x dx$$

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express $\sin^2 x$ via $\cos(2x)$

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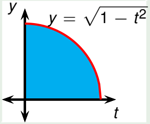
Example

Set $t = \cos x$, $x \in [0, \frac{\pi}{2}] \Rightarrow \sin x \geq 0$. Then $dt = d(\cos x) = -\sin x dx$.



$$\int_{t=0}^{t=1} \sqrt{1 - t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1 - \cos^2 x} \sin x dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} \sqrt{\sin^2 x} \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right) dx$$

express $\sin^2 x$ via $\cos(2x)$

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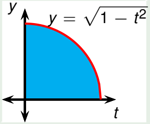

$$\int_{t=0}^{t=1} \sqrt{1-t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} \sqrt{\sin^2 x} \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{\pi}{4} \quad .$$

Example

$$\int \tan^8 x \sec^4 x \, dx$$

Example

$$\int \tan^8 x \sec^4 x \mathrm{d}x = \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x$$

Example

$$\int \tan^8 x \sec^4 x \, dx = \int \tan^8 x \sec^2 x \sec^2 x \, dx$$
$$= \int \tan^8 x \sec^2 x \, d(\textbf{?} \quad)$$

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\ &= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x)\end{aligned}$$

Example

$$\int \tan^8 x \sec^4 x dx = \int \tan^8 x \sec^2 x \sec^2 x dx$$
$$= \int \tan^8 x \sec^2 x d(\tan x)$$
$$= \int \tan^8 x \left(? \qquad \right) d(\tan x)$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \sec^2 x d(\tan x) \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x)\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx\\ &= \int \tan^8 x \sec^2 x d(\tan x)\\ &= \int \tan^8 x\left(1+\tan^2 x\right) d(\tan x)\\ &= \int u^8\left(1+u^2\right) du\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\
&= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) \mathrm{d}(\tan x) \\
&= \int u^8 \left(1 + u^2\right) \mathrm{d}u \\
&= \int \left(u^8 + u^{10}\right) \mathrm{d}u
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\int \tan^8 x \sec^4 x dx = \int \tan^8 x \sec^2 x \sec^2 x dx$$

$$= \int \tan^8 x \sec^2 x d(\tan x)$$

$$= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x)$$

$$= \int u^8 \left(1 + u^2\right) du$$

$$= \int \left(u^8 + u^{10}\right) du$$

$$= \textbf{?}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
&= \int \tan^8 x \sec^2 x d(\tan x) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\
&= \int u^8 \left(1 + u^2\right) du \\
&= \int \left(u^8 + u^{10}\right) du \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\int \tan^8 x \sec^4 x dx = \int \tan^8 x \sec^2 x \sec^2 x dx$$
$$= \int \tan^8 x \sec^2 x d(\tan x)$$
$$= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x)$$
$$= \int u^8 \left(1 + u^2\right) du$$
$$= \int \left(u^8 + u^{10}\right) du$$
$$= \frac{u^9}{9} + \frac{u^{11}}{11} + C$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \sec^2 x d(\tan x) \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\ &= \int u^8 \left(1 + u^2\right) du \\ &= \int \left(u^8 + u^{10}\right) du \\ &= \frac{u^9}{9} + \frac{u^{11}}{11} + C\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \sec^2 x d(\tan x) \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\ &= \int u^8 \left(1 + u^2\right) du \\ &= \int \left(u^8 + u^{10}\right) du \\ &= \frac{u^9}{9} + \frac{u^{11}}{11} + C \\ &= \frac{\tan^9 x}{9} + \frac{\tan^{11} x}{11} + C \quad .\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\int \tan^5 x \sec^9 x dx$$

Example

$$\int \tan^5 x \sec^9 x \, dx = \int \tan^4 x \sec^8 x \tan x \sec x \, dx$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\mathbf{?} \qquad)$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$
$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\ &= \int \tan^4 x \sec^8 x d(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$
\begin{aligned}
\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\
&= \int \tan^4 x \sec^8 x d(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 du \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 du
\end{aligned}
$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$
$$= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)$$

Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$
$$= \int \left(1 - 2u^2 + u^4\right) u^8 du$$
$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$
$$= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)$$

Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$
$$= \int \left(1 - 2u^2 + u^4\right) u^8 du$$
$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$
$$= \textbf{?}$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\ &= \int \tan^4 x \sec^8 x d(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(1 - u^2\right)^2 u^8 du \\ &= \int \left(1 - 2u^2 + u^4\right) u^8 du \\ &= \int \left(u^8 - 2u^{10} + u^{12}\right) du \\ &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx\\ &= \int \tan^4 x \sec^8 x d(\sec x)\\ &= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)\\ &= \int \left(1-u^2\right)^2 u^8 du\\ &= \int \left(1-2u^2+u^4\right) u^8 du\\ &= \int \left(u^8 - 2u^{10} + u^{12}\right) du\\ &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$
$$= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)$$

Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$
$$= \int \left(1 - 2u^2 + u^4\right) u^8 du$$
$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$
$$= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\ &= \int \tan^4 x \sec^8 x d(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(1 - u^2\right)^2 u^8 du \\ &= \int \left(1 - 2u^2 + u^4\right) u^8 du \\ &= \int \left(u^8 - 2u^{10} + u^{12}\right) du \\ &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$
\begin{aligned}
\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\
&= \int \tan^4 x \sec^8 x d(\sec x) && \text{Can we rewrite } \tan^4 x \text{ via } \sec x? \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) && \text{Set } u = \sec x \\
&= \int \left(1 - u^2\right)^2 u^8 du \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 du \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) du \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C \\
&= \frac{\sec^9 x}{9} - 2\frac{\sec^{11} x}{11} + \frac{\sec^{13} x}{13} + C \quad .
\end{aligned}
$$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx$$

$n -$ even, $n \geq 2$

$$\int \tan^m x \sec^n x dx$$

$m -$ odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$$

n – even, $n \geq 2$
$\sec^2 x dx$
$= d(\tan x)$

$$\int \tan^m x \sec^n x dx$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$$

$$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$$

n – even, $n \geq 2$
$\sec^2 x dx = d(\tan x)$
Express $\sec x$ via $\tan x$

$$\int \tan^m x \sec^n x dx$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$$
$$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$$

n – even, $n \geq 2$
$\sec^2 x dx = d(\tan x)$
Express $\sec x$ via $\tan x$

$$\int \tan^m x \sec^n x dx$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx$	m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx = d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx = d(\sec x)$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	$n -$ even, $n \geq 2$ $\sec^2 x dx = d(\tan x)$
$= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	$m -$ odd, $n \geq 1$ $\tan x \sec x dx = d(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$$

$$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$$

$$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$$

n – even, $n \geq 2$
$\sec^2 x dx = d(\tan x)$
Express $\sec x$ via $\tan x$
Set $u = \tan x$

$$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$$

$$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$$

m – odd, $n \geq 1$
$\tan x \sec x dx = d(\sec x)$
Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$; $\sec^2 x dx = d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$; $\tan x \sec x dx = d(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx = d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx = d(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$
$= \int \left(u^2 - 1\right)^{\frac{m-1}{2}} u^n du$	Set $u = \sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x)\\ &= \int \tan^m x\left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)\\ &= \int u^m\left(1+u^2\right)^{\frac{n-2}{2}} du\end{aligned}$$

n – even, $n \geq 2$

$\sec^2 x dx = d(\tan x)$

Express $\sec x$ via $\tan x$

Set $u = \tan x$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)\\ &= \int \left(u^2-1\right)^{\frac{m-1}{2}} u^n du\end{aligned}$$

m – odd, $n \geq 1$

$\tan x \sec x dx = d(\sec x)$

Express $\tan x$ via $\sec x$

Set $u = \sec x$

Outside of the above cases we either use more tricks or resort to the general method $x = 2\arctan t$.

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx = d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx = d(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$
$= \int \left(u^2 - 1\right)^{\frac{m-1}{2}} u^n du$	Set $u = \sec x$

Outside of the above cases we either use more tricks or resort to the general method $x = 2 \arctan t$.

Example

$$\int \tan x \mathrm{d}x$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(\textbf{?} \qquad)$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = \int \frac{1}{\cos x} \mathrm{d}(-\cos x)$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \qquad \Big| \text{ Set } u = \cos x$$

$$= -\int \frac{\mathrm{d}u}{u}$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x$$

$$= -\int \frac{\mathrm{d}u}{u}$$

Example

Example

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} d(-\cos x) \quad \Big| \text{ Set } u = \cos x$$
$$= -\int \frac{du}{u} = -\ln|u| + C$$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C\end{aligned}$$

Set $u = \cos x$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Bigg| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$.

Example

$$\int \sec x \mathrm{d}x$$

Example

$$\begin{aligned}\int \tan x\mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\int \sec x\mathrm{d}x = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Bigg| \text{ Set } u = \cos x \\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \qquad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x}\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + {\color{red}\sec x \tan x}}{\sec x + \tan x}{\color{red}\mathrm{d}x}\\ &= \int \frac{{\color{red}\mathrm{d}}(\tan x + {\color{red}\sec x})}{\sec x + \tan x}\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

Set $u = \cos x$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x \\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x} \\ &= \int \frac{\mathrm{d}u}{u}\end{aligned}$$

Set $u = \sec x + \tan x$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

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Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

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Example

$$\int \tan^3 x dx$$

Example

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

Example

$$\int \tan^3 x dx = \int \tan x \tan^2 x dx$$
$$= \int \tan x \left(\sec^2 x - 1 \right) dx$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1 \right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx
\end{aligned}$$

Example

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\ &= \int \tan x \left(\sec^2 x - 1 \right) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx\end{aligned}$$

Example

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Example

$$
\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1 \right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \textbf{?}
\end{aligned}
$$

Example

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\ &= \int \tan x \left(\sec^2 x - 1 \right) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x d(\tan x) - \textbf{?}\end{aligned}$$

Example

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\ &= \int \tan x \left(\sec^2 x - 1 \right) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x d(\tan x) - \ln|\sec x|\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1 \right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \ln|\sec x| \qquad \Big| \text{ Set } u = \tan x \\
&= \int u du + \ln\left|\frac{1}{\sec x}\right|
\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
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&= \int u du + \ln \left| \frac{1}{\sec x} \right| \\
&= \frac{u^2}{2} + \ln |\cos x| + C
\end{aligned}$$

Example

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&= \int u du + \ln \left| \frac{1}{\sec x} \right| \\
&= \frac{u^2}{2} + \ln|\cos x| + C \\
&= \frac{\tan^2 x}{2} + \ln|\cos x| + C
\end{aligned}$$

Example

$$\int \sec^3 x \mathrm{d}x$$

Example

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

Example

$$\int \sec^3 x \mathrm{d}x = \int \sec x \sec^2 x \mathrm{d}x$$
$$= \int \sec x \mathrm{d}(\mathbf{?} \quad)$$

Example

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\ &= \int \sec x d(\tan x)\end{aligned}$$

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x)\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x)\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\ &= \sec x \tan x - \int \tan \quad x \textbf{?} \qquad \mathrm{d}x\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx\\ &= \int \sec x d(\tan x)\\ &= \sec x \tan x - \int \tan x d(\sec x)\\ &= \sec x \tan x - \int \tan \quad x \sec x \tan x dx\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\ &= \int \sec x d(\tan x) \\ &= \sec x \tan x - \int \tan x d(\sec x) \\ &= \sec x \tan x - \int \tan\ \ x \sec x \tan x dx\end{aligned}$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx
\end{aligned}
$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\ &= \int \sec x d(\tan x) \\ &= \sec x \tan x - \int \tan x d(\sec x) \\ &= \sec x \tan x - \int \tan^2 x \sec x dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx\end{aligned}$$

| Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\ &= \int \sec x d(\tan x) \\ &= \sec x \tan x - \int \tan x d(\sec x) \\ &= \sec x \tan x - \int \tan^2 x \sec x dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx\end{aligned}$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx
\end{aligned}
$$

Integrate by parts

Example

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Integrate by parts

Example

$$
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\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
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&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2\int \sec^3 x dx &= \sec x \tan x + \textbf{?} \qquad\qquad + C
\end{aligned}
$$

Integrate by parts

Example

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2 \int \sec^3 x dx &= \sec x \tan x + \textbf{?} \qquad + C
\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
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&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2\int \sec^3 x dx &= \sec x \tan x + \ln|\sec x + \tan x| + C
\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\
\int \sec^3 x dx &= \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + K.
\end{aligned}$$

Integrate by parts

To evaluate integrals of the form

1. $\int \sin mx \cos nx dx$
2. $\int \sin mx \sin nx dx$
3. $\int \cos mx \cos nx dx$

use the corresponding identity:

1. $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Example

$$\int \sin 4x \cos 5x \mathrm{d}x$$

Example

$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2}[\sin(4x - 5x) + \sin(4x + 5x)] \, dx$$

Example

$$\begin{aligned}\int \sin 4x \cos 5x \mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\ &= \frac{1}{2}\int (\sin(-x)+\sin(9x))\mathrm{d}x\end{aligned}$$

Example

$$\begin{aligned}\int \sin 4x \cos 5x \mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\ &= \frac{1}{2}\int (\sin(-x)+\sin(9x))\mathrm{d}x \\ &= \frac{1}{2}\int (-\sin x+\sin(9x))\mathrm{d}x\end{aligned}$$

Example

$$\begin{aligned}\int \sin 4x \cos 5x \mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\ &= \frac{1}{2}\int(\sin(-x)+\sin(9x))\mathrm{d}x \\ &= \frac{1}{2}\int(-\sin x+\sin(9x))\mathrm{d}x \\ &= \frac{1}{2}(\cos x-\frac{1}{9}\cos(9x))+C\end{aligned}$$