Math 141
Lecture 7

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Outline

1. Integrals of form $\int R(x, \sqrt{ax^2+bx+c})dx$, R - rational function
   - Transforming to the forms $\sqrt{x^2+1}, \sqrt{-x^2+1}, \sqrt{x^2-1}$
   - Table of Euler and trig substitutions
   - The case $\sqrt{x^2+1}$
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   - The case $\sqrt{x^2-1}$
2. Rationalizing Substitutions

Integrals of form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$, R - rational function

Let $R(x, y)$ be an arbitrary rational expression in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\displaystyle\int R\left(x, \sqrt{ax^2+bx+c}\right)\mathrm{d}x$*?*

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  - Use trigonometric substitution or Euler substitution to transform to trigonometric or rational function integral (no radicals).

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- We motivate why we need such integrals by examples such as computing the area of an ellipse.

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$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} =$$

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- With $x = a\sin\theta$, the old variable is a function of the new one.

Linear substitutions to simplify radicals $\sqrt{ay^2+by+c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2+by+c}$, $a \neq 0$, $b^2-4ac \neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2+1}$
  - $\sqrt{-x^2+1}$
  - $\sqrt{x^2-1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

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Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\sqrt{x^2+x+1} = \sqrt{x^2+2\frac{1}{2}x + \quad ? - \quad ? + 1}$$

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Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\sqrt{x^2+x+1} = \sqrt{x^2 + 2\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}$$

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Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\begin{aligned}\sqrt{x^2+x+1} &= \sqrt{x^2+2\frac{1}{2}x+\frac{1}{4} \quad -\frac{1}{4} \quad +1} \\ &= \sqrt{\left(x+ \quad \color{red}{?}\right)^2 + \color{red}{?}}\end{aligned}$$

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$$\begin{aligned}\sqrt{x^2+x+1} &= \sqrt{x^2+2\frac{1}{2}x+\frac{1}{4} \quad -\frac{1}{4} \quad +1} \\ &= \sqrt{\left(x+\frac{1}{2} \quad\right)^2 + \quad \frac{3}{4}} \\ &= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}\end{aligned}$$

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\begin{aligned}
\sqrt{x^2+x+1} &= \sqrt{x^2+2\frac{1}{2}x+\frac{1}{4} \quad -\frac{1}{4} \quad +1} \\
&= \sqrt{\left(x+\frac{1}{2}\quad\right)^2 + \quad \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}
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Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

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\sqrt{x^2+x+1} &= \sqrt{x^2 + 2\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \\
&= \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2 + 1} \\
&= \frac{\sqrt{3}}{2}\sqrt{u^2+1},
\end{aligned}$$

where $u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.

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Example

Use linear subst. to transform $\sqrt{-2x^2+x+1}$ to multiple of $\sqrt{-u^2+1}$.

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Example

Use linear subst. to transform $\sqrt{-2x^2+x+1}$ to multiple of $\sqrt{-u^2+1}$.

$$\sqrt{-2x^2+x+1} = \sqrt{-2\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)}$$

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Use linear subst. to transform $\sqrt{-2x^2+x+1}$ to multiple of $\sqrt{-u^2+1}$.

$$\begin{aligned}\sqrt{-2x^2+x+1} &= \sqrt{-2\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)} \\ &= \sqrt{-2\left(x^2 - 2\frac{1}{4}x + \textbf{?} \quad - \textbf{?} \quad - \frac{1}{2}\right)}\end{aligned}$$

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&= \sqrt{-2\left(\left(x-\frac{1}{4}\right)^2-\frac{9}{16}\right)} \\
&= \sqrt{\frac{9}{8}\left(-\frac{16}{9}\left(x-\frac{1}{4}\right)^2+1\right)}
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- So far, with linear transformations we converted all integrals of the form $\displaystyle\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$ to one of the three forms: $\int R(x, \sqrt{x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{-x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{x^2-1})\mathrm{d}x$.

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- Each of the above integrals can be transformed to a rational trigonometric integral using 3 pairs of substitutions: $x = \tan\theta$, $x = \cot\theta$; $x = \sin\theta$, $x = \cos\theta$; $x = \csc\theta$, $x = \sec\theta$.

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$x = \tan\theta$, $x = \cot\theta$; $x = \sin\theta$, $x = \cos\theta$; $x = \csc\theta$, $x = \sec\theta$.

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- The Euler substitutions directly transform the integral to a rational function integral.
- We will demonstrate that the Euler substitutions are rational.

Trigonometric substitution and Euler substitution

Expression	Substitution	Variable range	Relevant identity
$\sqrt{x^2+1}$	$x = \tan\theta$ $x = \cot\theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\theta \in (0, \pi)$	$1+\tan^2\theta = \sec^2\theta$ $1+\cot^2\theta = \csc^2\theta$
$\sqrt{-x^2+1}$	$x = \sin\theta$ $x = \cos\theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\theta \in (0, \pi)$	$1-\sin^2\theta = \cos^2\theta$ $1-\cos^2\theta = \cos^2\theta$
$\sqrt{x^2-1}$	$x = \csc\theta$ $x = \sec\theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\csc^2\theta - 1 = \cot^2\theta$ $\sec^2\theta - 1 = \tan^2\theta$
Euler substitution by applying in addition $\theta = 2\arctan t$			
$\sqrt{x^2+1}$	$x = \frac{2t}{1-t^2}$ $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$	$-1 < t < 1$ $0 < t$	(?) (?)
$\sqrt{-x^2+1}$	$x = \frac{2t}{1+t^2}$ $x = \frac{1-t^2}{1+t^2}$	$-1 \leq t \leq 1$ $0 < t$	(?) (?)
$\sqrt{x^2-1}$	$x = \frac{1}{2}\left(\frac{1}{t} + t\right)$ $x = \frac{1+t^2}{1-t^2}$	$t \in (-\infty, -1) \cup [0, 1)$ $t \in (-\infty, -1) \cup [0, 1)$	(?) (?)

Trigonometric substitution $x = \cot\theta$ for $\sqrt{x^2+1}$

The trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$:

$$\sqrt{x^2+1} \quad =$$

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The trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$:

$$\sqrt{x^2+1} = \sqrt{\cot^2\theta + 1}$$

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The trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$:

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\cot^2\theta + 1} \\ &= \sqrt{\frac{\cos^2\theta}{\sin^2\theta} + 1}\end{aligned}$$

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&= \sqrt{\frac{1}{\sin^2\theta}} = \frac{1}{\sqrt{\sin^2\theta}} \\
&= \frac{1}{\sin\theta}
\end{aligned}$$

when $\theta \in (0, \pi)$ we have $\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

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$$\begin{aligned} x &= \cot\theta \\ \sqrt{x^2+1} &= \frac{1}{\sin\theta} = \csc\theta \\ \mathrm{d}x &= \qquad\qquad\qquad \mathbf{?}\mathrm{d}\theta \\ \theta &= \operatorname{arccot} x \quad . \end{aligned}$$

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Example

$$\int \frac{1}{x^2\sqrt{x^2 + 9}}\mathrm{d}x$$

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$$\int \frac{1}{x^2\sqrt{x^2+9}}dx = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}dx$$

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$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

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$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$ $\csc\theta > 0$

Example

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\ &= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\ &= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\ &= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\ &= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta\end{aligned}$$

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Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int\frac{1}{\cos^2\theta}\mathrm{d}(\mathbf{?}\qquad)
\end{aligned}
$$

Set

$$
\begin{aligned}
\frac{x}{3} &= \cot\theta \\
x &= 3\cot\theta
\end{aligned}
$$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Example

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)
\end{aligned}$$

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&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2}
\end{aligned}$$

Set

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$\theta \in (0, \pi)$

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Set $u = \cos\theta$

Example

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$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\Big(-3\csc^2\theta\Big)\mathrm{d}\theta$$

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$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = \textbf{?} \qquad + C$$

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\end{aligned}$$

Set
$\frac{x}{3} = \cot\theta$
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\end{aligned}$$

Set
$$\frac{x}{3} = \cot\theta$$
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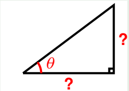
Set $u = \cos\theta$



Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}dx = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}dx$$
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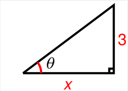

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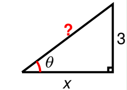

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$$
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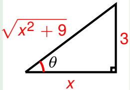

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}dx = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}dx$$

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$$= \frac{1}{9}\int \frac{du}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C$$

$$= -\frac{\sqrt{x^2+9}}{9x} + C$$

Set

$$\frac{x}{3} = \cot\theta$$
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$\theta \in (0, \pi)$

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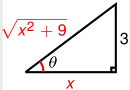
Set $u = \cos\theta$



Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t$, transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t, \ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x \quad =$$

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What if we compose the above? We get the Euler substitution:

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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&= \frac{1-t^2}{2t}
\end{aligned}$$

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $dx, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $d\theta, \cos\theta, \sin\theta$ to rational form.

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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} =$$

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We can furthermore compute

$$\sqrt{x^2+1} = \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2 + 1}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\begin{aligned} \sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t} - t\right)^2 + 1} \\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t} {\color{red}+} t\right)^2} \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2 + 1}\\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2} \qquad \left|\ \begin{array}{l}\sqrt{\left(\frac{1}{t}+t\right)^2} = \frac{1}{t}+t\\ \text{because } t > 0\end{array}\right.\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2+1}\\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2}\\ &= \frac{1}{2}\left(\frac{1}{t}+t\right) \quad .\end{aligned}$$

$\sqrt{\left(\frac{1}{t}+t\right)^2} = \frac{1}{t}+t$ because $t > 0$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2+1} \\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2} \qquad \left|\ \begin{array}{l}\sqrt{\left(\frac{1}{t}+t\right)^2} = \frac{1}{t}+t \\ \text{because } t>0\end{array}\right. \\ &= \color{red}{\frac{1}{2}\left(\frac{1}{t}+t\right)} \quad .\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) .$$

Finally compute

$$\mathrm{d}x =$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right)$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) =$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) .$$

Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) \end{aligned} \quad .$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) \quad . \end{aligned}$$

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $dx, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $d\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} dx &= d\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)dt \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$ is given by:

$$\begin{aligned} x &= \frac{1}{2}\left(\frac{1}{t} - t\right), \qquad t > 0 \\ \sqrt{x^2+1} &= \frac{1}{2}\left(\frac{1}{t} + t\right) \\ \mathrm{d}x &= -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \sqrt{x^2+1} - x \quad . \end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\mathrm{d}x \quad =$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $dx = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)dt$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}dx &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)dt\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)dt\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $dx = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)dt$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}dx &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)dt \\ &= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)dt \\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C
\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $dx = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)dt$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}dx &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)dt \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)dt \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C \\
&= \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1}-x)^2}-\left(\sqrt{x^2+1}-x\right)^2\right) \\
&\quad -\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C
\end{aligned}
$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1}-x)^2} - \left(\sqrt{x^2+1}-x\right)^2\right) \\
&\quad -\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right) + C
\end{aligned}
$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $dx = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)dt$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}dx &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)dt \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)dt \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1}-x)^2} - \left(\sqrt{x^2+1}-x\right)^2\right) \\
&\quad -\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right) + C
\end{aligned}
$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1} - x)^2} - \left(\sqrt{x^2+1} - x\right)^2\right) \\
&\quad - \frac{1}{2}\ln\left(\sqrt{x^2+1} - x\right) + C
\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t}+t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C \\
&= \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1}-x)^2} - \left(\sqrt{x^2+1}-x\right)^2\right) \\
&\quad -\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C
\end{aligned}$$

Recall Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\mathrm{d}x = \frac{1}{8}\left(\frac{1}{(\sqrt{x^2+1}-x)^2} - \left(\sqrt{x^2+1}-x\right)^2\right) - \frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right) + C$$

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&= \left(\sqrt{x^2+1}+x\right)^2 - \left(\sqrt{x^2+1}-x\right)^2 = 4x\sqrt{x^2+1}
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$$\ln\left(\frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}\right) =$$

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Example

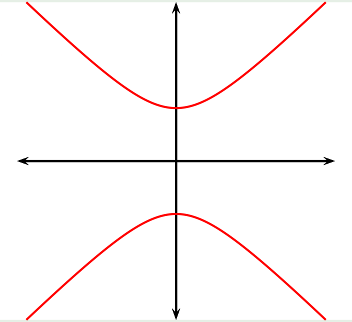
$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= \frac{1}{2}x\sqrt{x^2+1} \\ &\quad + \frac{1}{2}\ln\left(\sqrt{x^2+1} + x\right) + C\end{aligned}$$

The last expression can be transformed to:

$$\begin{aligned}\ln\left(\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right) &= \ln\left(\frac{1}{\sqrt{x^2+1}+x}\right) \\ &= -\ln\left(\sqrt{x^2+1}+x\right)\end{aligned}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



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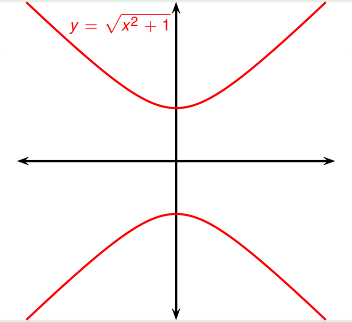

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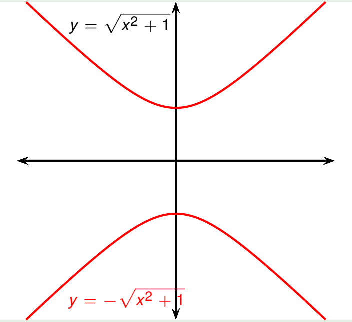

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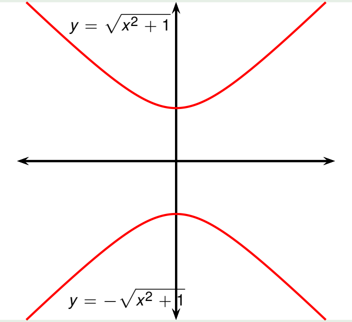

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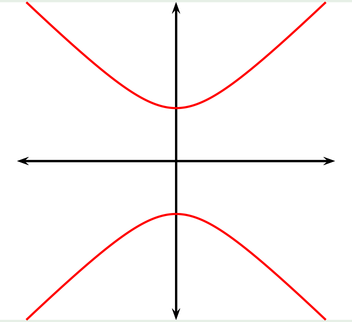

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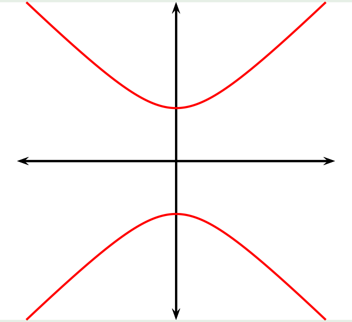

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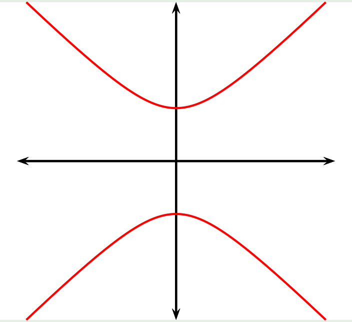

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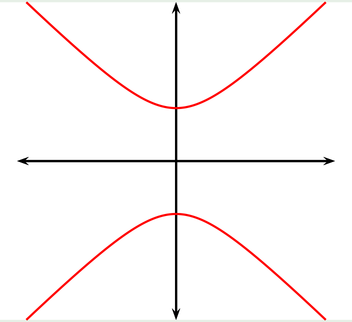

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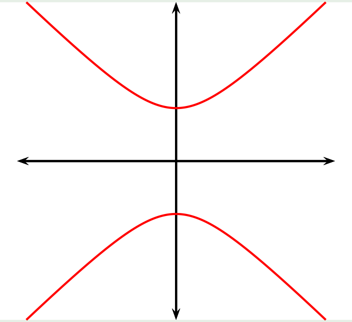

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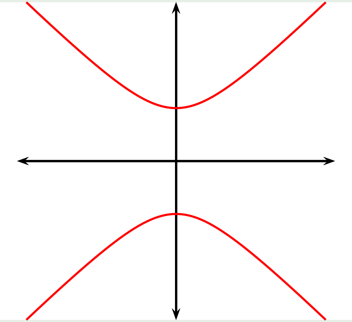

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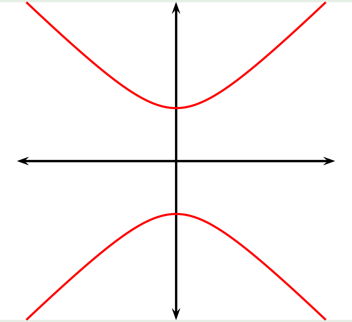

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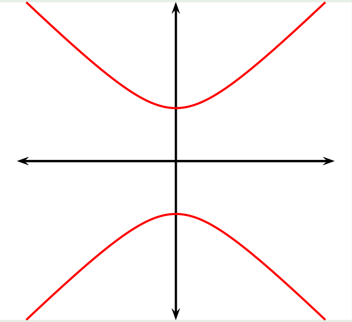

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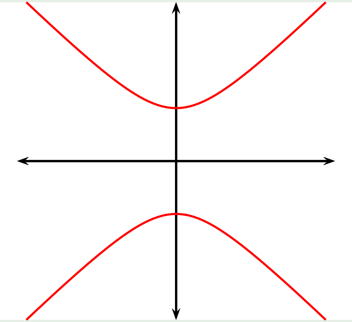

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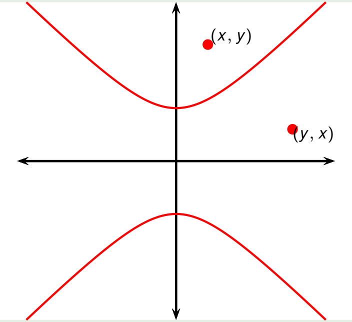

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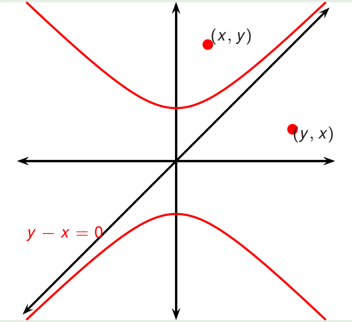

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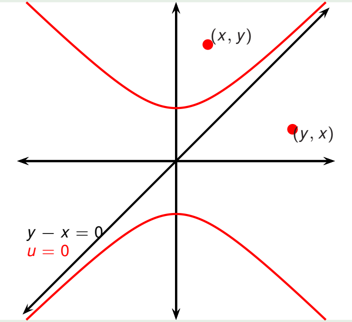

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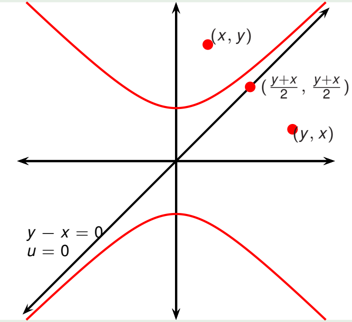

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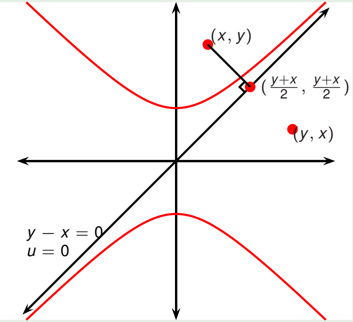

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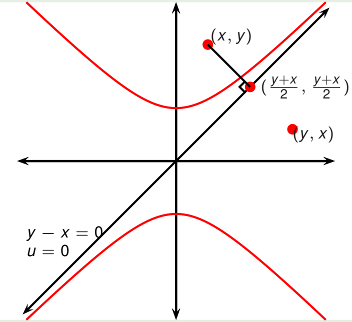

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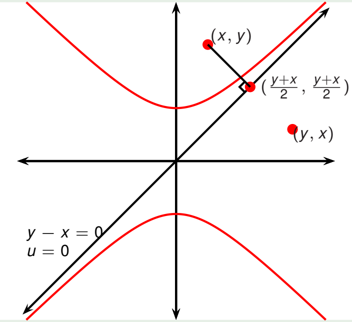

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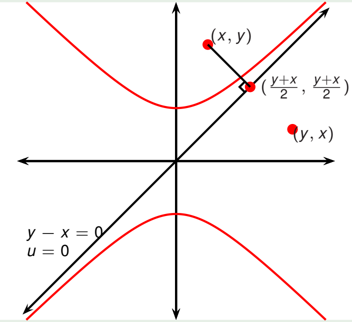

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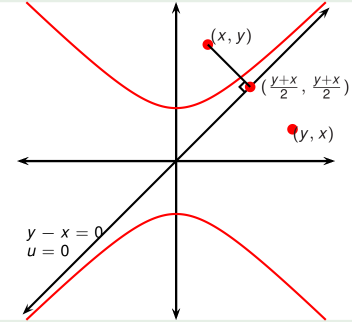

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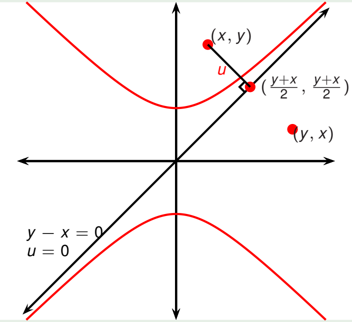

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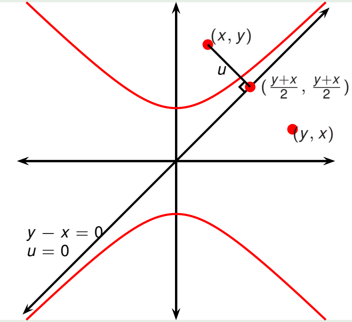

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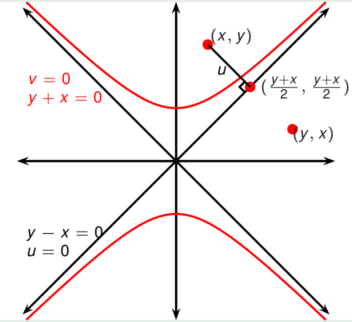

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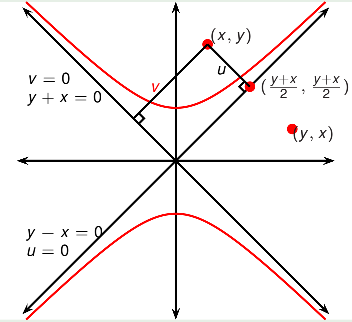

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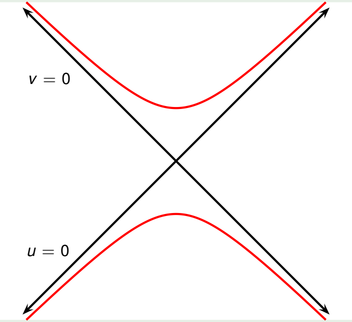

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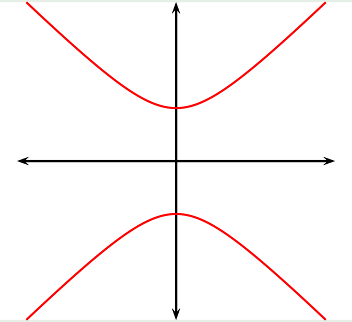

The area in question is:

$$\int_{?}^{?} 2\sqrt{x^2+1}dx$$

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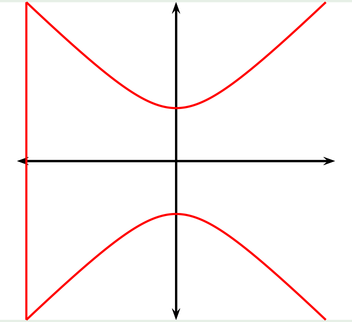

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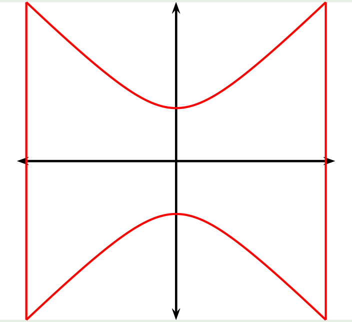

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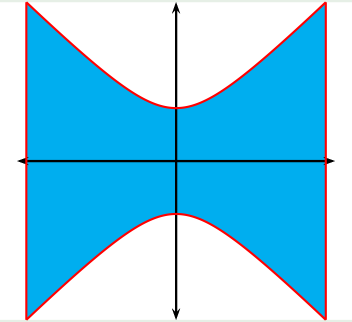

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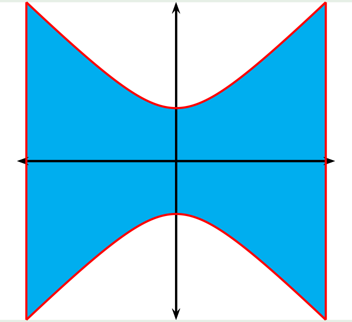

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$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}dx$$

$$= \left[x\sqrt{x^2+1}\right.$$

$$\left.\ln\left(\sqrt{x^2+1}+x\right)\right]_{-2\sqrt{2}}^{2\sqrt{2}}$$

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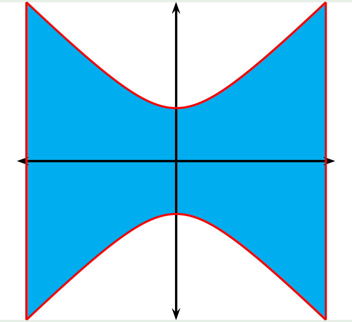

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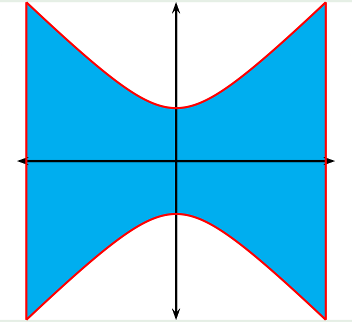

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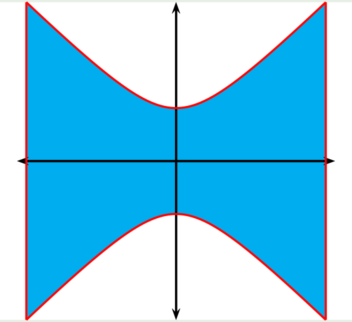

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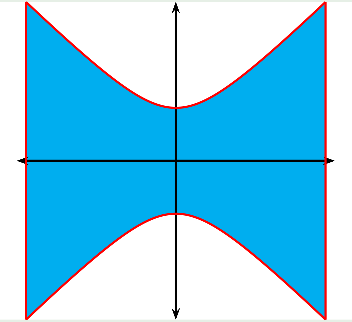
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1}\right.$$

$$\left.+\ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals u. Similarly compute that signed distance b-n (x, y) and the line $v = 0$ equals v. $\Rightarrow y^2 - x^2 = 1$ is the hyperbola $v = \frac{1/2}{v}$ in the (u, v)-plane.

The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}dx$$

$$= 2\left[x\sqrt{x^2+1}\right.$$

$$\left.\ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1}\right.$$

$$\left.+\ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

$$\approx 20.496$$

Example

Find $\int \frac{x}{\sqrt{x^2+4}}dx$.

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Trigonometric substitution $x = \cos\theta$ for $\sqrt{-x^2+1}$

The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$:

$$\sqrt{-x^2+1} \quad =$$

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| when $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have $\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

To summarize:

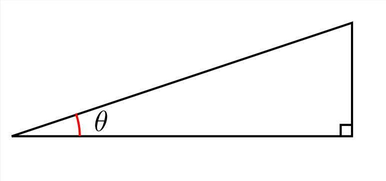
Definition

The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$ is given by:

$$\begin{aligned} x &= \cos\theta \\ \sqrt{-x^2+1} &= \sin\theta \\ dx &= -\sin\theta d\theta \\ \theta &= \arccos x \quad . \end{aligned}$$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} dx$.



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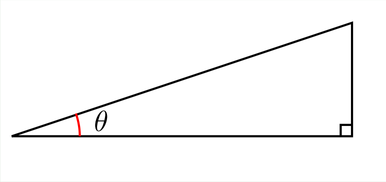

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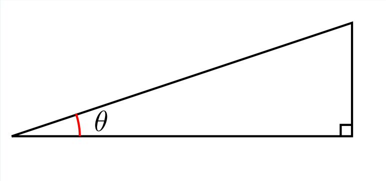

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Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$.

- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
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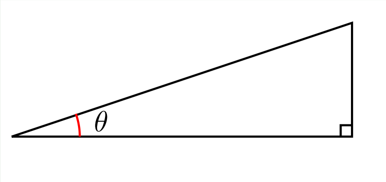

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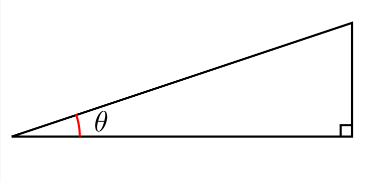

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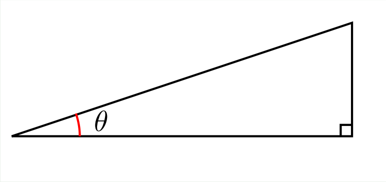

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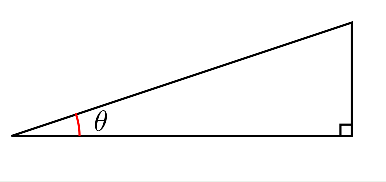

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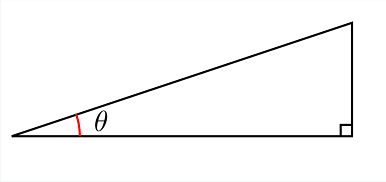

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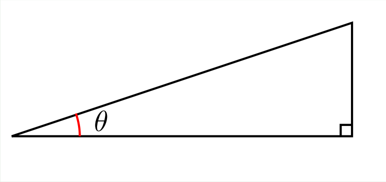

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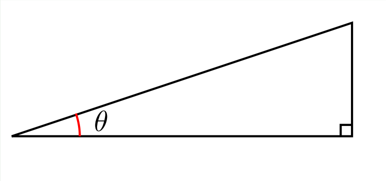

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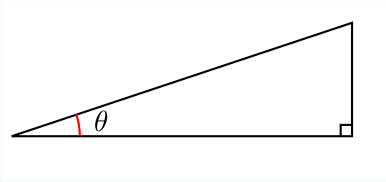

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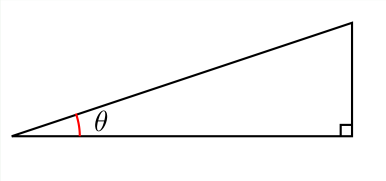

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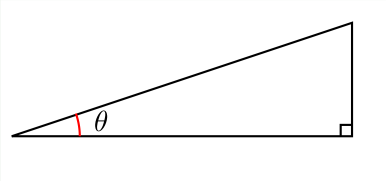

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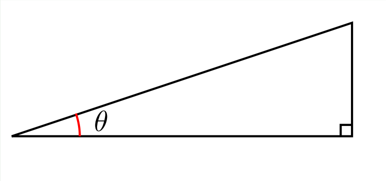

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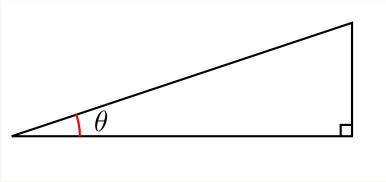

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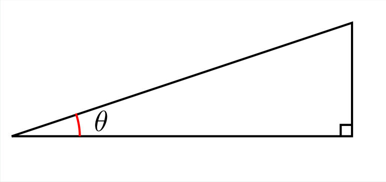

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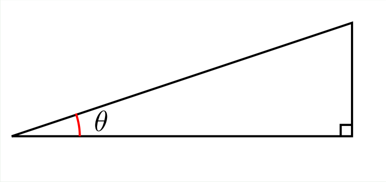

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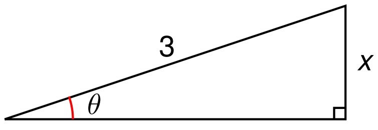

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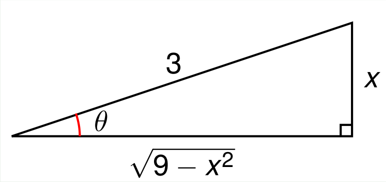

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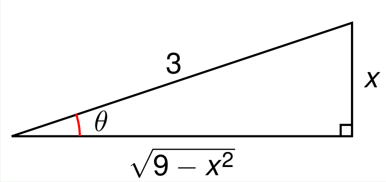

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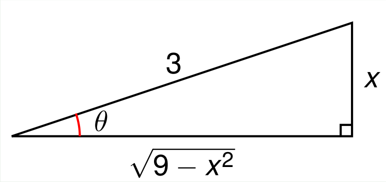

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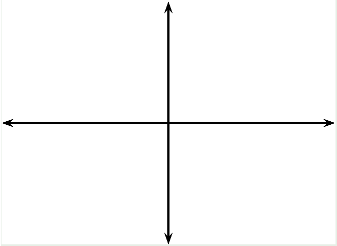
&= \int (\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



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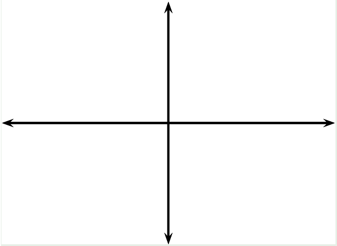

Express y via x:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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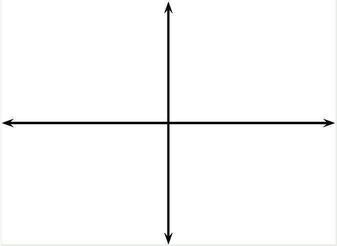

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$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

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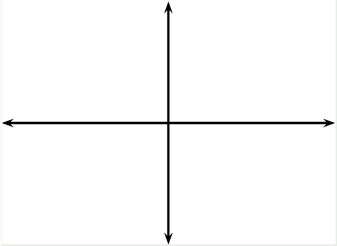
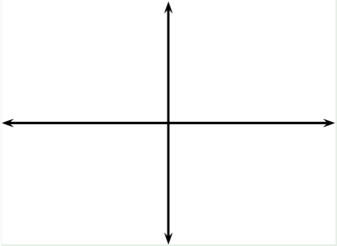


Express y via x:

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\
\frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\
y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



Express y via x:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \\ y^2 &= b^2\left(1 - \frac{x^2}{a^2}\right) \\ y &= \pm b\sqrt{1 - \frac{x^2}{a^2}} \end{aligned}$$

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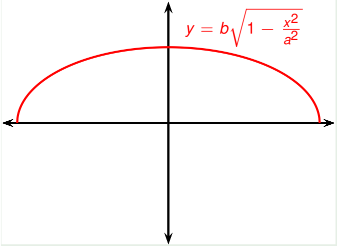

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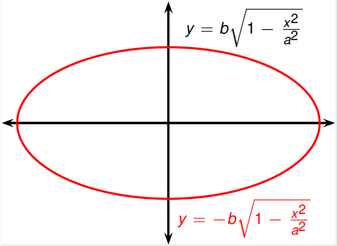

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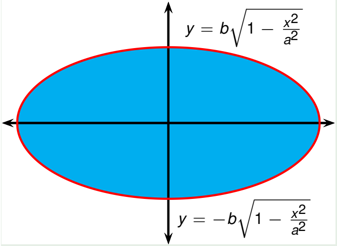

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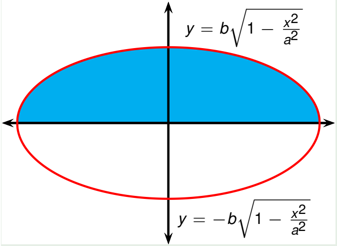

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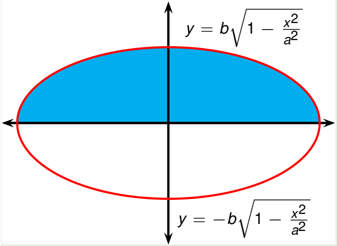

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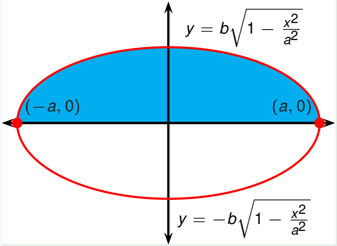

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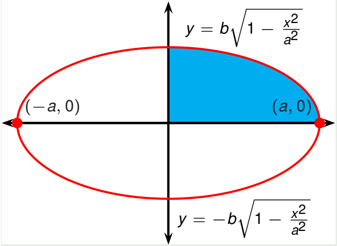

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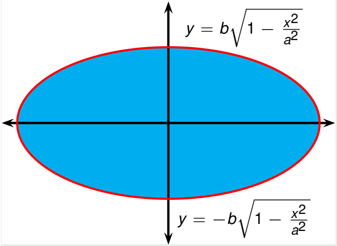

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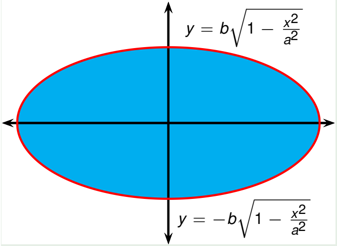

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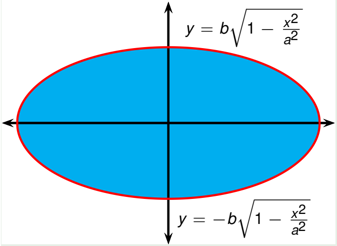

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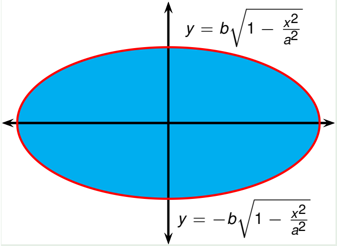

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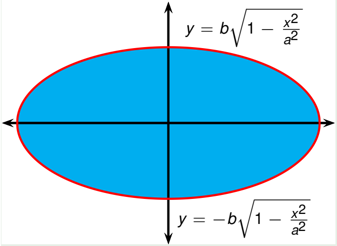

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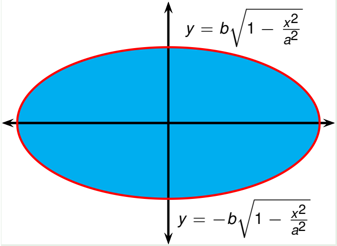

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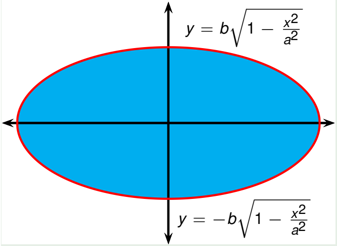

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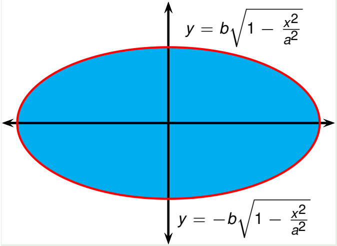

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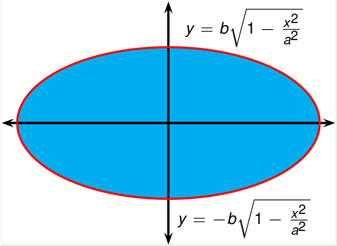

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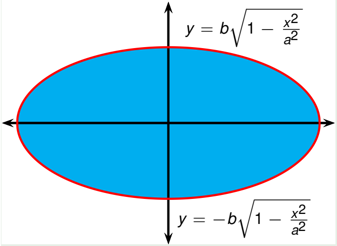

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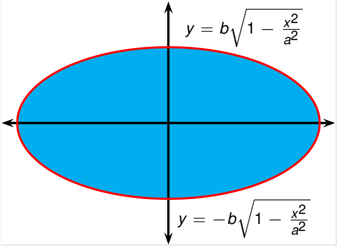

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&= a\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta)+1}{2}\mathrm{d}\theta \\
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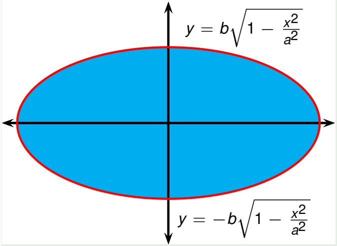

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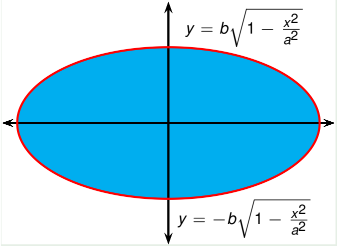

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&= a\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= a\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta) + 1}{2} d\theta \\
&= a\left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= a\left(0 + \frac{\pi}{4} - (0 + 0)\right)
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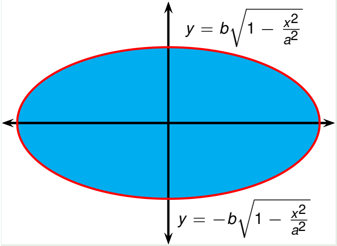

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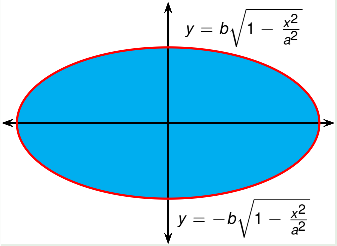
&= a\left(0 + \frac{\pi}{4} - (0+0)\right) \\
&= \frac{a\pi}{4}
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



The area in question is

$$\int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}dx$$

$$= 4\int_{0}^{a} b\sqrt{1 - \frac{x^2}{a^2}}dx$$

$$= 4b\frac{a\pi}{4}$$

Trig subst.: set $x = a\sin\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$.
Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^a \sqrt{1 - \frac{x^2}{a^2}}dx &= \int_0^{\frac{\pi}{2}} \cos\theta \, d\left(a\sin\theta\right) \\
&= a\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= a\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta) + 1}{2} d\theta \\
&= a\left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= a\left(0 + \frac{\pi}{4} - (0+0)\right) \\
&= \frac{a\pi}{4}
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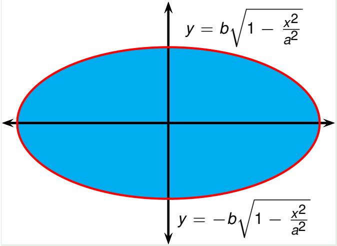

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- $\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{x}{\sqrt{4-(x+1)^2}}dx = \int \frac{u-1}{\sqrt{4-u^2}}du$
- Let $u = 2\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $du = 2\cos\theta d\theta$.
- $\sqrt{4-u^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2|\cos\theta| = 2\cos\theta$

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{u-1}{\sqrt{4-u^2}}du = \int \frac{2\sin\theta-1}{2\cos\theta}2\cos\theta d\theta$$
$$= \int(2\sin\theta-1)d\theta = -2\cos\theta-\theta+C$$
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Example

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$.

- Complete the square under the root sign:
- $3-2x-x^2 = 3+1-(x^2+2x+1) = 4-(x+1)^2$
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$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}dx &= \int \frac{u-1}{\sqrt{4-u^2}}du = \int \frac{2\sin\theta-1}{2\cos\theta}2\cos\theta d\theta\\ &= \int(2\sin\theta-1)d\theta = -2\cos\theta-\theta+C\\ &= -\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C\end{aligned}$$

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Example

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$.

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\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $dx, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t$, transforms $d\theta, \cos\theta, \sin\theta$ to rational form.

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- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x =$$

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$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \end{aligned}$$

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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

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- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= \cos\theta \\
&= \cos(2\arctan t) \\
&= \frac{1 - \tan^2(\arctan t)}{1+\tan^2(\arctan t)}
\end{aligned}
$$

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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \\ &= \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} \\ &= \frac{1-t^2}{1+t^2} \end{aligned}$$

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Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

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What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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$$\sqrt{-x^2+1} = \sqrt{1 - \left(\frac{1-t^2}{1+t^2}\right)^2} = \sqrt{\frac{(1+t^2)^2 - (1-t^2)^2}{(1+t^2)^2}}$$

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Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

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&= \sqrt{\frac{4t^2}{(1+t^2)^2}} && \left|\ \sqrt{4t^2} = 2t \text{ because } t > 0\right.
\end{aligned}$$

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- $x = \cos\theta$ transforms $dx, x, \sqrt{-x^2+1}$ to trig form.
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Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

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(1+t^2)x &= 1-t^2 \\
t^2(x+1) &= 1-x \\
t^2 &= \frac{1-x}{1+x} \\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}} && \text{here we use } t > 0
\end{aligned}$$

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(1+t^2)x &= 1-t^2\\
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\end{aligned}$$

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(1+t^2)x &= 1-t^2\\
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\end{aligned}$$

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(1+t^2)x &= 1-t^2\\
t^2(x+1) &= 1-x\\
t^2 &= \frac{1-x}{1+x}\\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{\sqrt{-x^2+1}}{x+1} \qquad \text{here we use } t>0
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$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

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Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

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$$\mathrm{d}x$$

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Definition

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Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad =$$

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when $\theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ we have $\tan\theta \geq 0$ and so $\sqrt{\tan^2\theta} = \tan\theta$

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Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2 + 1}$ is given by:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\ \sqrt{x^2 - 1} &= \tan\theta \\ dx &= \qquad\qquad\qquad \mathbf{?}d\theta \\ \theta &= \operatorname{arcsec} x \quad . \end{aligned}$$

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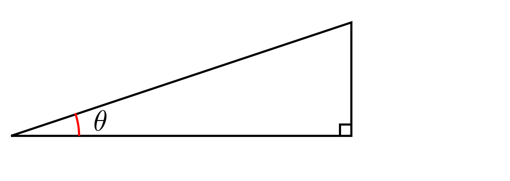
\sqrt{x^2-1} &= \tan\theta \\
\mathrm{d}x &= \frac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \sec\theta\tan\theta \;\; \mathrm{d}\theta \\
\theta &= \operatorname{arcsec} x \quad .
\end{aligned}$$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



Example

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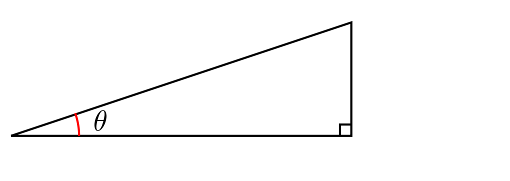
- $x =$
- $\mathrm{d}x =$

$\sqrt{x^2 - a^2} =$



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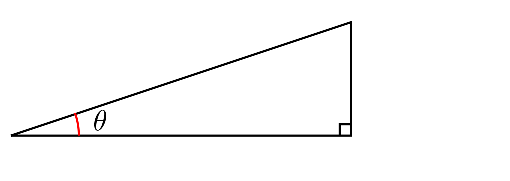

Example

Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.

- $x = a\sec\theta$,
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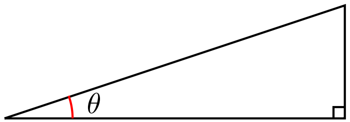

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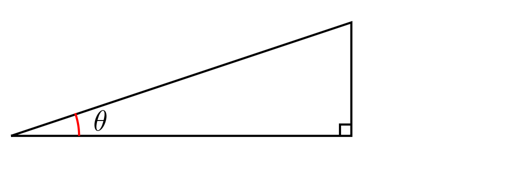

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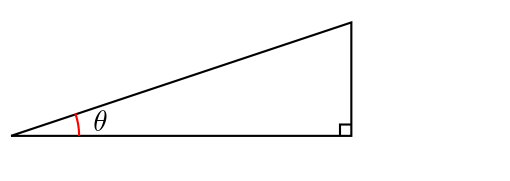

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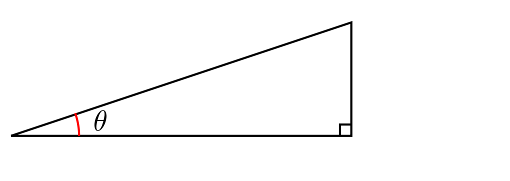

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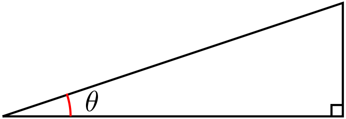

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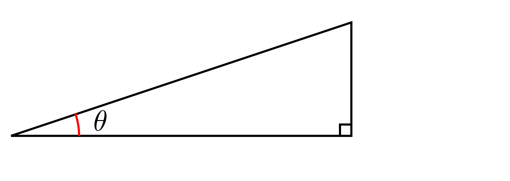

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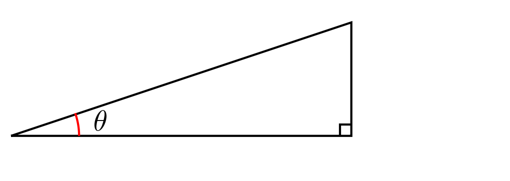

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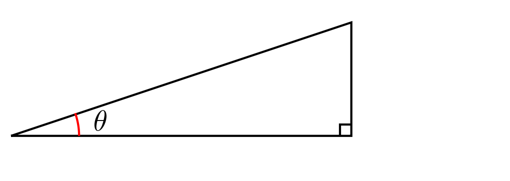

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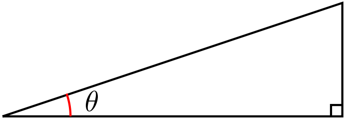

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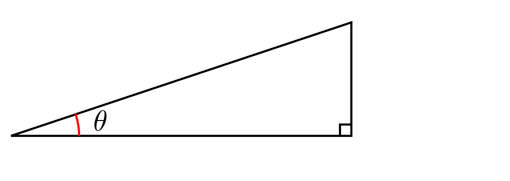

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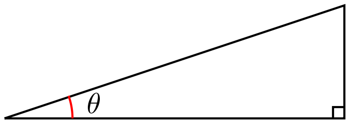

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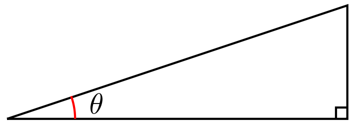

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$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Example

Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.

- $x = a\sec\theta$,
  $0 < \theta < \pi/2$ or
  $\pi < \theta < 3\pi/2$.
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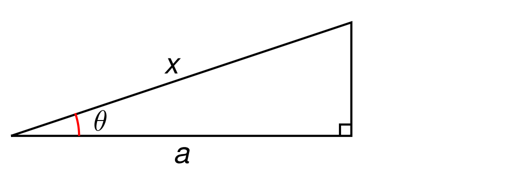

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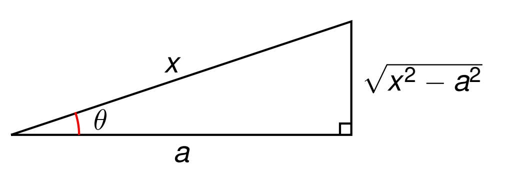

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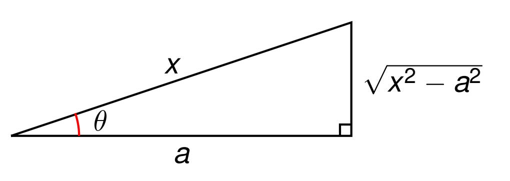

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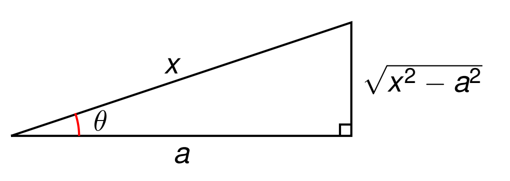

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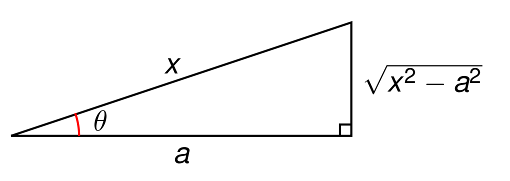

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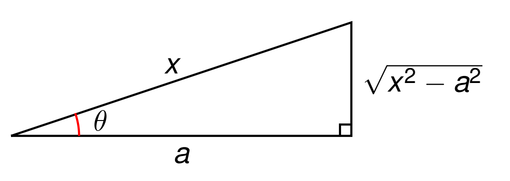

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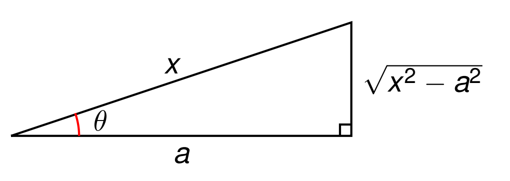

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Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.

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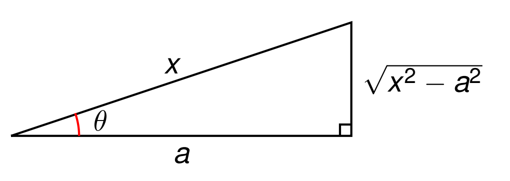

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&= \ln\left|x + \sqrt{x^2-a^2}\right| + C_1
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

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$$x =$$

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What if we compose the above? We get the Euler substitution:

$$x = \sec\theta = \frac{1}{\cos\theta}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \end{aligned}$$

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$\cos(2z) =$

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$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \\ &= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

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What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1}$$

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$$\begin{aligned} \sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\ &= \sqrt{\frac{(1+t^2)^2 - (1-t^2)^2}{(1-t^2)^2}} \end{aligned}$$

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$$\sqrt{x^2-1} = \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1}$$

$$= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} \quad \Big| \ (1+t^2)^2-(1-t^2)^2 = \mathbf{?}$$

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\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} && \Big|\ (1+t^2)^2-(1-t^2)^2 = 4t^2 \\
&= \sqrt{\frac{4t^2}{(1-t^2)^2}} && \Big|\ t, 1-t^2 \text{ have same sign when } t \in (-\infty,-1)\cup[0,1) \\
&= \frac{2t}{1-t^2}
\end{aligned}$$

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$$(1-t^2)x = 1+t^2$$

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x &= \frac{1+t^2}{1-t^2}\\
(1-t^2)x &= 1+t^2\\
(1+x)t^2 &= x-1\\
t^2 &= \frac{x-1}{x+1}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $dx, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, $t \in (-\infty, -1) \cup [0, 1)$ rationalizes $d\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$\begin{aligned}
x &= \frac{1+t^2}{1-t^2}\\
(1-t^2)x &= 1+t^2\\
(1+x)t^2 &= x-1\\
t^2 &= \frac{x-1}{x+1}\\
t &= \pm\sqrt{\frac{x-1}{x+1}}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$= \;\; ?dt$$

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- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$\mathrm{d}x = \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right) = \frac{4t}{(1-t^2)^2}\,\mathrm{d}t$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2-1}$ corresponding to $x = \sec\theta$ is given by:

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2}, \qquad t \in (-\infty, -1) \cup [0, 1) \\ \sqrt{x^2-1} &= \frac{2t}{1-t^2} \\ dx &= \frac{4t}{(1-t^2)^2}dt \\ t &= \pm\frac{\sqrt{x^2-1}}{x+1} \quad . \end{aligned}$$

Rationalizing Substitutions

Some nonrational fractions can be changed into rational fractions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x =$ and $\mathrm{d}x =$.

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x = \int \frac{\quad}{\quad}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} dx$.

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x =$ and $dx =$.

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x =$ and $\mathrm{d}x =$.

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x = \int \frac{u}{}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} dx$.

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx =$.

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2 - 4}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x =$.

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x \quad = \quad \int \frac{u}{u^2 - 4}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} dx$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u du$.

$$\int \frac{\sqrt{x+4}}{x} dx \quad = \quad \int \frac{u}{u^2 - 4} 2u du$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x = \int \frac{u}{u^2-4} 2u\mathrm{d}u = 2\int \frac{u^2}{u^2-4} \mathrm{d}u$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u = 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\ &= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u\end{aligned}$$

(long division)

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x} \mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u = 2\int \frac{u^2}{u^2-4} \mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right) \mathrm{d}u = 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&\quad \text{(long division)}
\end{aligned}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x}\mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4}2u\mathrm{d}u \;=\; 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1+\frac{4}{u^2-4}\right)\mathrm{d}u \;=\; 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&\quad \text{(long division)} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{1}{4}\cdot\frac{1}{u-2} - \frac{1}{4}\cdot\frac{1}{u+2}\right)\mathrm{d}u \\
&\quad \text{(partial fractions)}
\end{aligned}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x}\mathrm{d}x$.
Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}
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&\quad \text{(long division)} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{1}{4}\cdot\frac{1}{u-2} - \frac{1}{4}\cdot\frac{1}{u+2}\right)\mathrm{d}u \\
&\quad \text{(partial fractions)} \\
&= 2u + 2(\ln|u-2| - \ln|u+2|) + C
\end{aligned}$$

Example (Example 9, p. 517)

Find $\int \frac{\sqrt{x+4}}{x} dx$.

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u du$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x} dx &= \int \frac{u}{u^2-4} 2u du = 2 \int \frac{u^2}{u^2-4} du \\
&= 2 \int \left(1 + \frac{4}{u^2-4}\right) du = 2 \int du + 8 \int \frac{du}{u^2-4} \\
&\quad \text{(long division)} \\
&= 2 \int du + 8 \int \left(\frac{1}{4} \cdot \frac{1}{u-2} - \frac{1}{4} \cdot \frac{1}{u+2}\right) du \\
&\quad \text{(partial fractions)} \\
&= 2u + 2(\ln|u-2| - \ln|u+2|) + C \\
&= 2\sqrt{x+4} + 2 \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C
\end{aligned}$$