Math 141

Lecture 9

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Spring 2015

Outline

1. Improper Integrals
   - Type I: Infinite Intervals
   - Type II: Discontinuous Integrands
   - A Comparison Test for Improper Integrals

Improper Integrals

- The definition of $\int_a^b f(x)\mathrm{d}x$, where f is defined on $[a, b]$, has two requirements:
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Definition (Improper Integral)

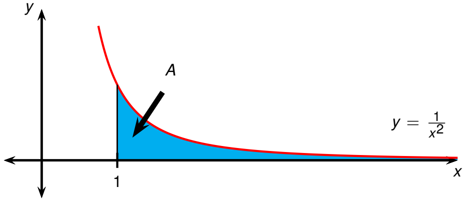
The integral

$$\int_a^b f(x)\mathrm{d}x$$

is called improper if one or more of the endpoints a and b is infinite, or if f has an infinite discontinuity on $[a,b]$.

Type I: Infinite Intervals

- Consider the region A that lies under $y = 1/x^2$, above the x-axis, and to the right of $x = 1$.



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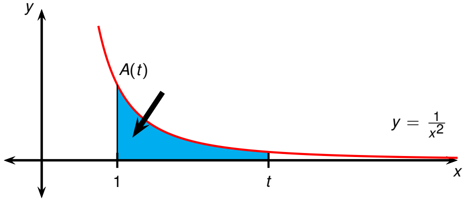

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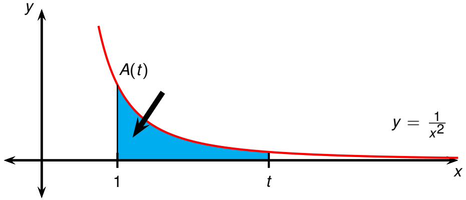

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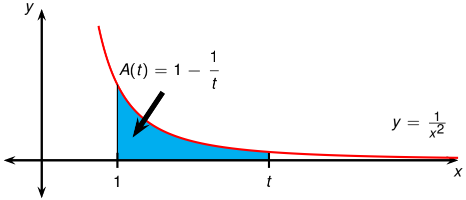

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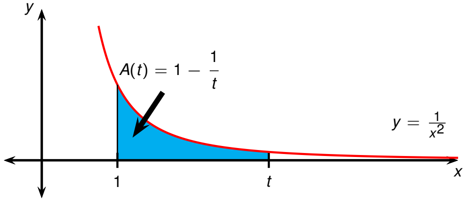

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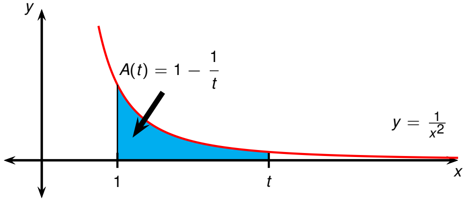

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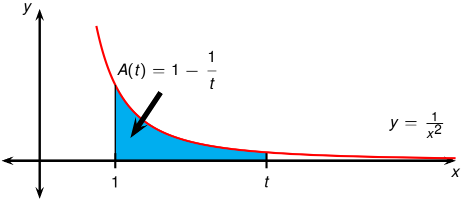

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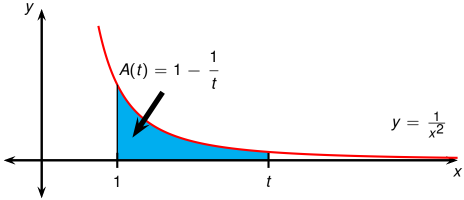

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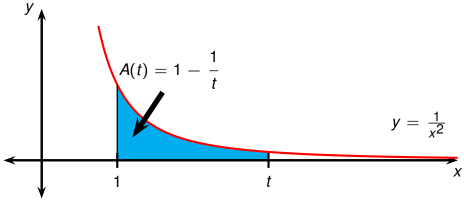

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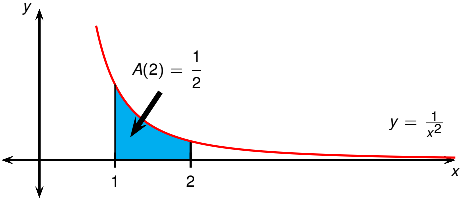

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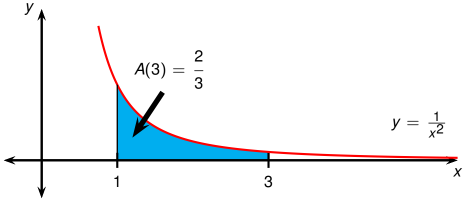

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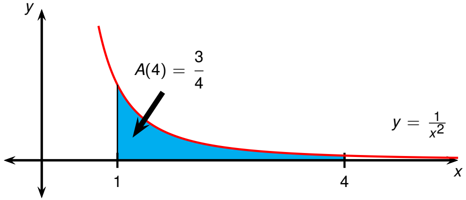

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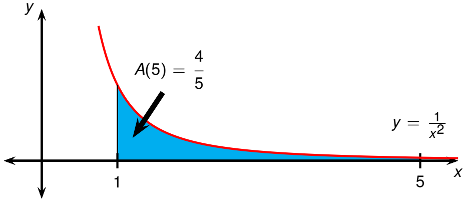

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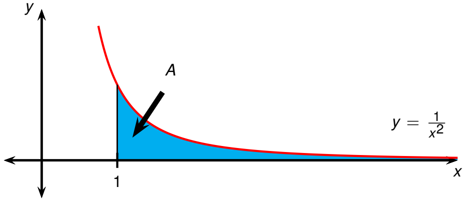

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- Also notice $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1$.
- We say that the area A is equal to 1 and write $\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = 1$.

Definition (Improper Integral of Type I)

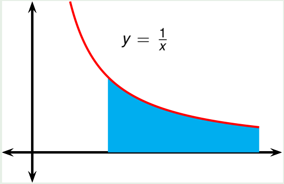
1. If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, then
$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\mathrm{d}x$$
if the limit exists.
2. If $\int_t^b f(x)\mathrm{d}x$ exists for every $t \leq b$, then
$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\mathrm{d}x$$
if the limit exists.

$\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are called convergent if the corresponding limit exists and divergent if it doesn't exist.

3. If both $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^a f(x)\mathrm{d}x$ are convergent, then we define
$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

Example

Determine whether $\int_1^\infty \frac{1}{x} dx$ is convergent or divergent.



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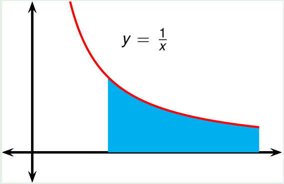

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx$$

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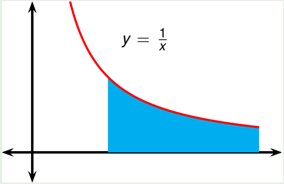

$$\int_1^\infty \frac{1}{x}\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{x}\mathrm{d}x$$

$$= \lim_{t\to\infty} [\ln x]_1^t$$

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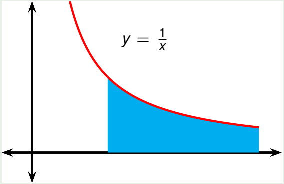

$$\begin{aligned} \int_1^\infty \frac{1}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t\to\infty} [\ln x]_1^t \\ &= \lim_{t\to\infty} (\ln t - \ln 1) \end{aligned}$$

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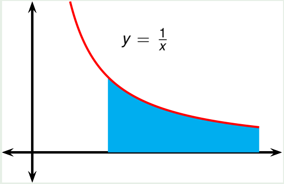

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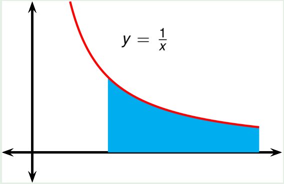

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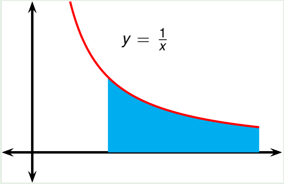

Infinite area

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}dx \\ &= \lim_{t\to\infty}[\ln x]_1^t \\ &= \lim_{t\to\infty}(\ln t - \ln 1) \\ &= \lim_{t\to\infty}\ln t = \infty\end{aligned}$$

Therefore the improper integral is divergent.

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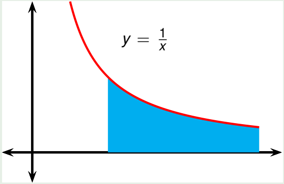

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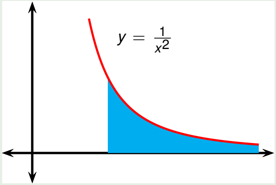

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$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$$

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Evaluate the two integrals separately:

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$$\int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} \mathrm{d}x$$

$$\int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} \mathrm{d}x$$

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\int_{-\infty}^{0} \frac{1}{1+x^2} dx &= \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} dx = \lim_{t \to -\infty} [\arctan x]_t^0 \\
&= \lim_{t \to -\infty} (\arctan 0 - \arctan t) = \lim_{t \to -\infty} (0 - \arctan t) \\
&= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\int_{0}^{\infty} \frac{1}{1+x^2} dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} dx$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$$

Evaluate the two integrals separately:

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\int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to-\infty} \left[\arctan x\right]_t^0 \\
&= \lim_{t\to-\infty} (\arctan 0 - \arctan t) = \lim_{t\to-\infty} (0 - \arctan t) \\
&= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to\infty} \left[\arctan x\right]_0^t$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

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&= \lim_{t\to\infty} (\arctan t - \arctan 0)
\end{aligned}$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$$

Evaluate the two integrals separately:

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\int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to-\infty} [\arctan x]_t^0 \\
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&= \lim_{t\to\infty} (\arctan t - \arctan 0) = \lim_{t\to\infty} \arctan t
\end{aligned}$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

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&= \lim_{t\to\infty} (\arctan t - \arctan 0) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}
\end{aligned}$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

Evaluate the two integrals separately:

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&= \lim_{t\to-\infty} (\arctan 0 - \arctan t) = \lim_{t\to-\infty} (0 - \arctan t) \\
&= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \\
\int_{0}^{\infty} \frac{1}{1+x^2} dx &= \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} dx = \lim_{t\to\infty} [\arctan x]_0^t \\
&= \lim_{t\to\infty} (\arctan t - \arctan 0) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}
\end{aligned}$$

Therefore $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

Example

For what values of p is the integral $\int_1^\infty \frac{1}{x^p} dx$ convergent?

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$$\int_1^\infty \frac{1}{x^p} \mathrm{d}x = \lim_{t \to \infty} \int_1^t \frac{1}{x^p} \mathrm{d}x$$

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$$\int_1^\infty \frac{1}{x^p}\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\mathrm{d}x = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

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- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.

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- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
- Therefore $\int_1^\infty \frac{1}{x^p} dx = \frac{1}{p-1}$ if $p > 1$, and so the integral is convergent.

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For what values of p is the integral $\int_1^\infty \frac{1}{x^p} dx$ convergent?

- We know from Example 1 that if $p = 1$, the integral is divergent.
- Assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \frac{\frac{1}{t^{p-1}} - 1}{1-p}$$

- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
- Therefore $\int_1^\infty \frac{1}{x^p} dx = \frac{1}{p-1}$ if $p > 1$, and so the integral is convergent.
- If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.

Example

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}dx$ convergent?

- We know from Example 1 that if $p = 1$, the integral is divergent.
- Assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\frac{\frac{1}{t^{p-1}}-1}{1-p}$$

- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
- Therefore $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$ if $p > 1$, and so the integral is convergent.
- If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.
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For what values of p is the integral $\int_1^\infty \frac{1}{x^p} dx$ convergent?

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- Assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t\to\infty} \frac{\frac{1}{t^{p-1}} - 1}{1-p}$$

- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
- Therefore $\int_1^\infty \frac{1}{x^p} dx = \frac{1}{p-1}$ if $p > 1$, and so the integral is convergent.
- If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.
- Therefore $\int_1^\infty \frac{1}{x^p} dx$ is divergent if $p < 1$.

Theorem

$\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$.

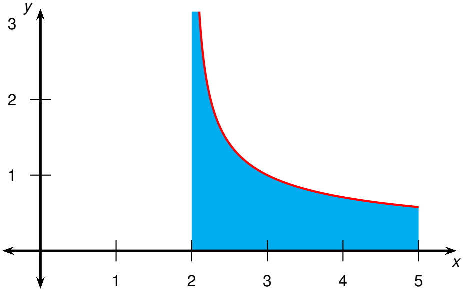
Type II: Discontinuous Integrands

We can use the same approach if the function f is discontinuous at one of the endpoints a and b in the integral $\int_a^b f(x)\mathrm{d}x$.

For example, $\frac{1}{\sqrt{x-2}}$ is discontinuous at 2, so we might wonder if the integral

$$\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$$

exists.



Definition (Improper Integral of Type II)

1. If f is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\mathrm{d}x$$
if the limit exists.

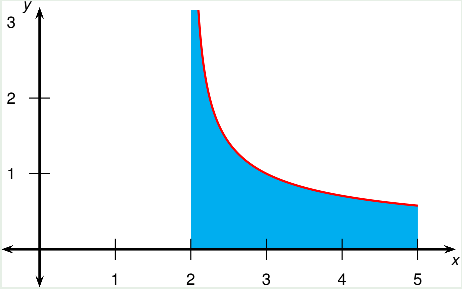
2. If f is continuous on $(a, b]$ and discontinuous at a, then
$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\mathrm{d}x$$
if the limit exists.

$\int_a^b f(x)\mathrm{d}x$ is called convergent if the corresponding limit exists and divergent if it doesn't exist.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent, then we define
$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

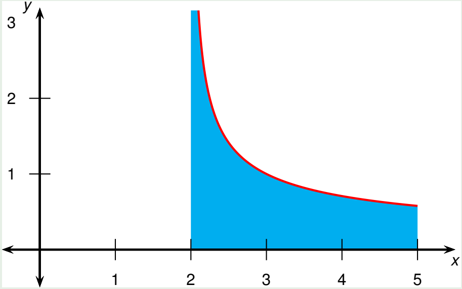
Example

Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.



Example

Find $\int_2^5 \frac{1}{\sqrt{x-2}} \mathrm{d}x$.
Observe that $x = 2$ is a vertical asymptote for the integrand.



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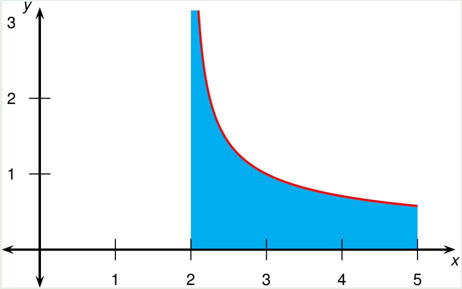

$$\int_2^5 \frac{1}{\sqrt{x-2}} \mathrm{d}x = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} \mathrm{d}x$$

Example

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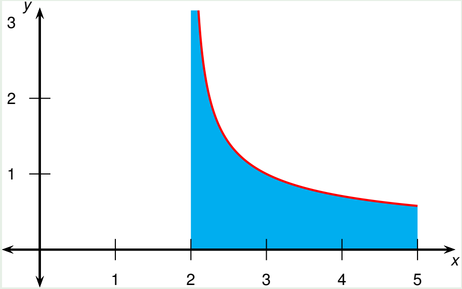
Observe that $x = 2$ is a vertical asymptote for the integrand.



$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\ = \quad & \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\ = \quad & \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5\end{aligned}$$

Example

Find $\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$.
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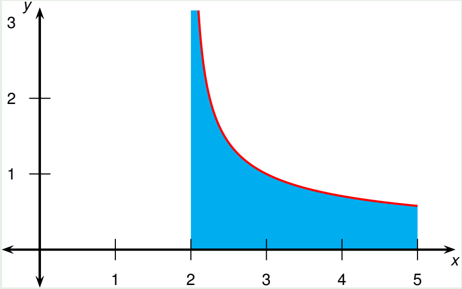
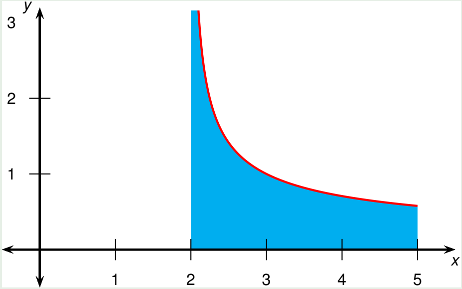


$$\begin{aligned}
&\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\
= &\lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\
= &\lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5 \\
= &\lim_{t\to 2^+} 2(\sqrt{5-2}-\sqrt{t-2})
\end{aligned}$$

Example

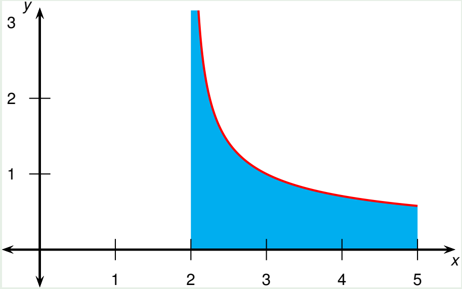
Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.
Observe that $x = 2$ is a vertical asymptote for the integrand.



$$
\begin{aligned}
& \int_2^5 \frac{1}{\sqrt{x-2}} dx \\
= & \lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} dx \\
= & \lim_{t \to 2^+} \left[2\sqrt{x-2} \right]_t^5 \\
= & \lim_{t \to 2^+} 2(\sqrt{5-2} - \sqrt{t-2}) \\
= & 2\sqrt{3}
\end{aligned}
$$

Example

Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.
Observe that $x = 2$ is a vertical asymptote for the integrand.



Area = $2\sqrt{3}$

$$\begin{aligned}
&\int_2^5 \frac{1}{\sqrt{x-2}} dx \\
= &\lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} dx \\
= &\lim_{t \to 2^+} \left[2\sqrt{x-2} \right]_t^5 \\
= &\lim_{t \to 2^+} 2(\sqrt{5-2} - \sqrt{t-2}) \\
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Example

Evaluate $\int_0^3 \frac{1}{x-1}\mathrm{d}x$.

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$$\int_0^3 \frac{1}{x-1}\mathrm{d}x = \int_0^1 \frac{1}{x-1}\mathrm{d}x + \int_1^3 \frac{1}{x-1}\mathrm{d}x$$

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Evaluate $\int_0^3 \frac{1}{x-1}\mathrm{d}x$.
Observe that $x = 1$ is a vertical asymptote for the integrand.

$$\int_0^3 \frac{1}{x-1}\mathrm{d}x = \int_0^1 \frac{1}{x-1}\mathrm{d}x + \int_1^3 \frac{1}{x-1}\mathrm{d}x$$

$$\int_0^1 \frac{\mathrm{d}x}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{\mathrm{d}x}{x-1}$$

Example

Evaluate $\int_0^3 \frac{1}{x-1} dx$.
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$$\int_0^1 \frac{dx}{x-1} \quad = \quad \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \left[\ln|x-1| \right]_0^t$$

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- Therefore the integral diverges.

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- Therefore the integral diverges.
- If we had not noticed the vertical asymptote, we might have made the following mistake:

$$\int_0^3 \frac{\mathrm{d}x}{x-1} = \left[\ln|x-1|\right]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

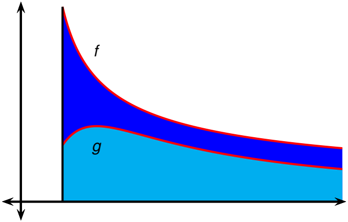
A Comparison Test for Improper Integrals

Sometimes it's impossible to find the exact value of an integral, but we still want to know if it's convergent or divergent. For such cases, we can sometimes use the following theorem.

Theorem (Comparison Theorem)

Suppose f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. *If $\int_a^\infty f(x)\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\mathrm{d}x$ is convergent.*
2. *If $\int_a^\infty g(x)\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\mathrm{d}x$ is divergent.*



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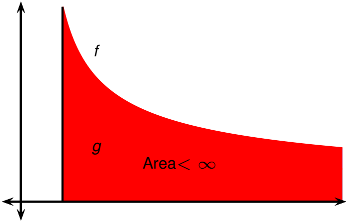

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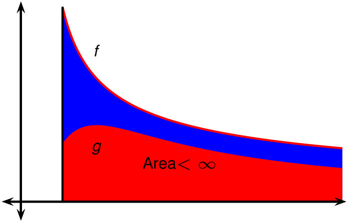

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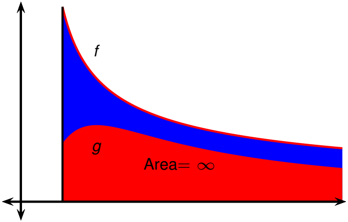

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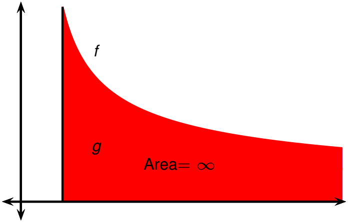

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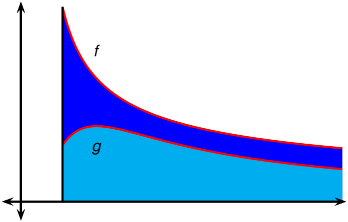

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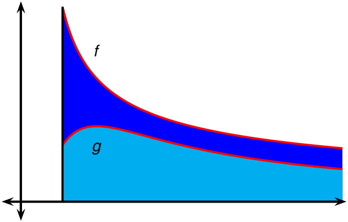

A similar theorem holds for Type II improper integrals.

Example

Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

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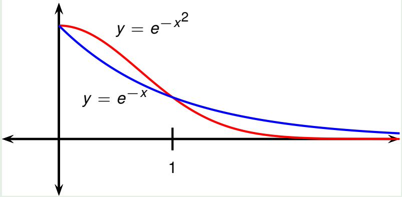

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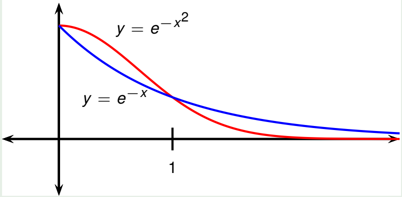

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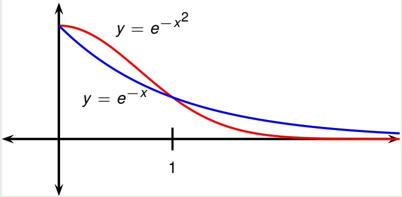

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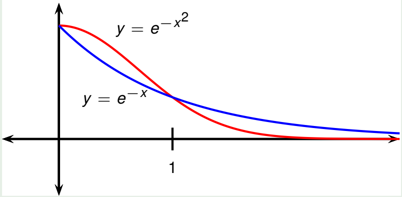

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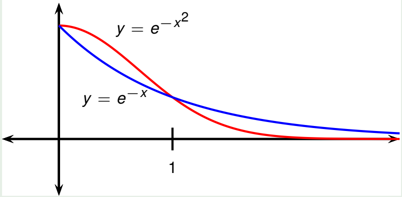

$$\int_1^\infty e^{-x} dx = \lim_{t\to\infty} \int_1^t e^{-x} dx = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t$$

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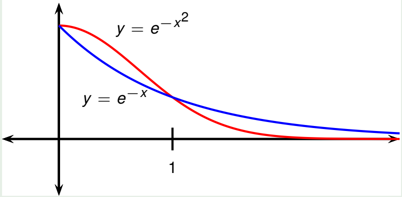

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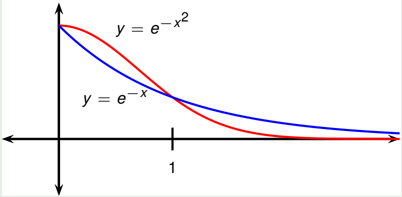

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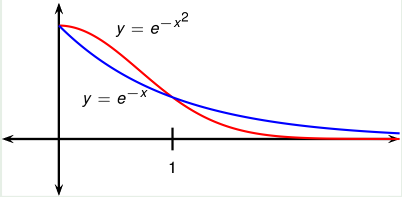

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Therefore by the Comparison Theorem, $\int_0^\infty e^{-x^2}dx$ converges.

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Is $\int_1^\infty \frac{1+e^{-x}}{x} \mathrm{d}x$ convergent or divergent?

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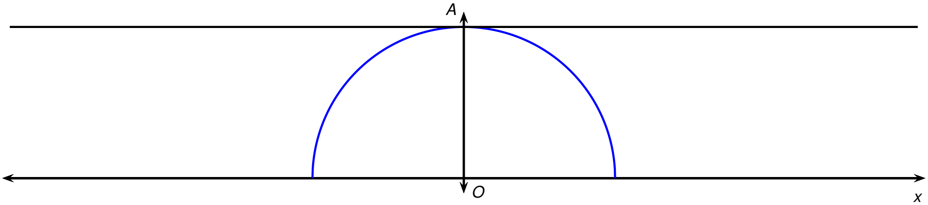
- Therefore $\int_1^\infty \frac{1+e^{-x}}{x} \mathrm{d}x$ is divergent by the Comparison Theorem.



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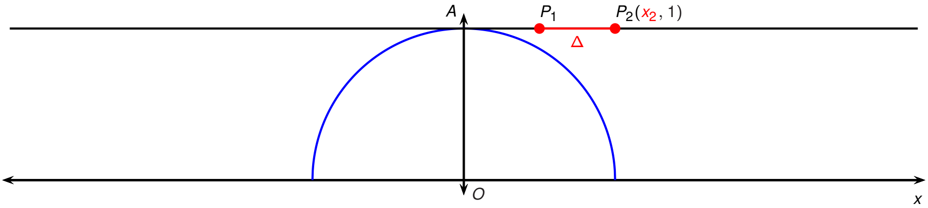

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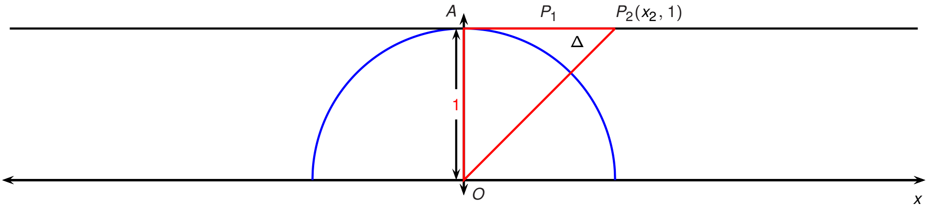

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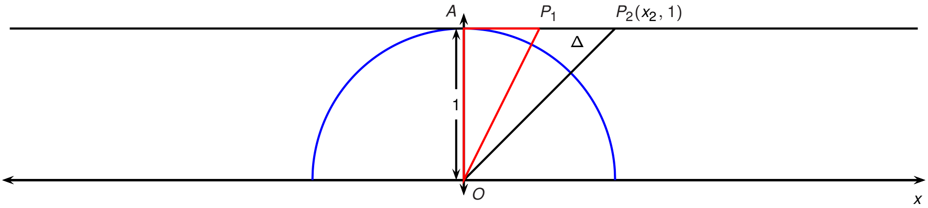

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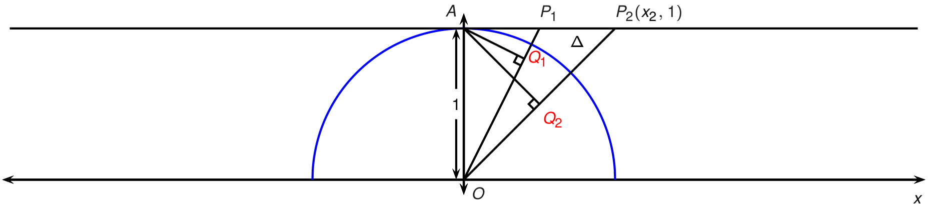

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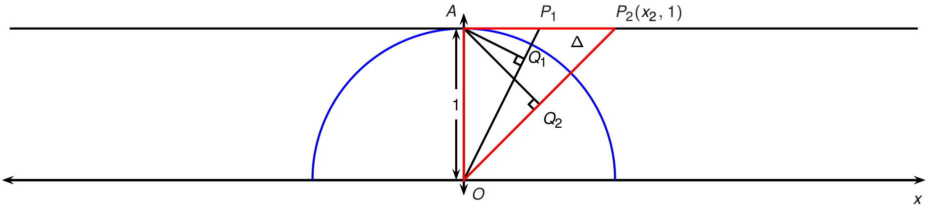

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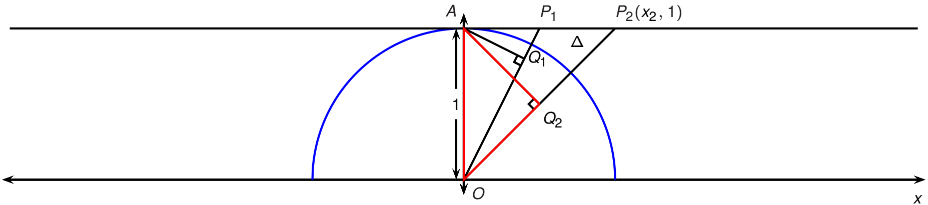

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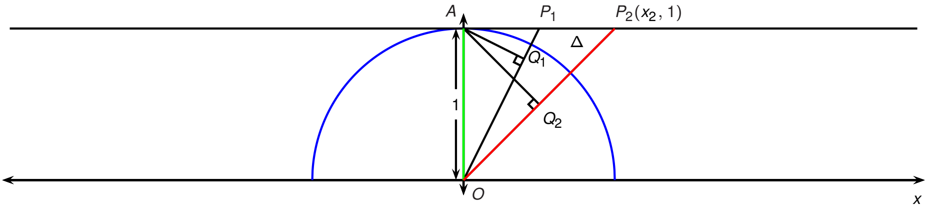

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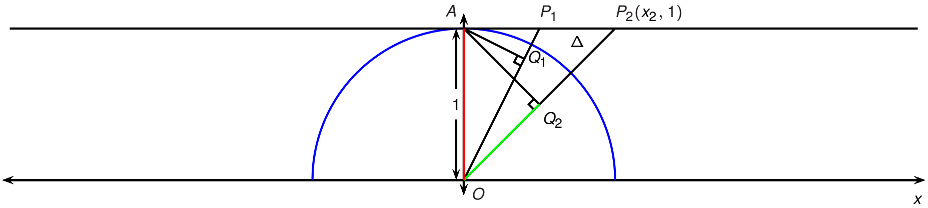

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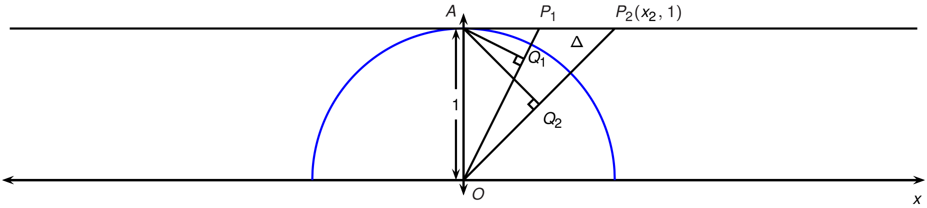

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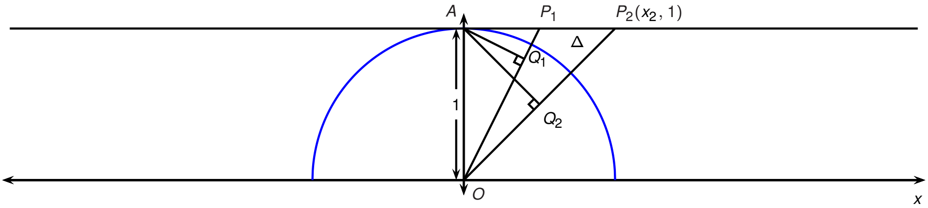

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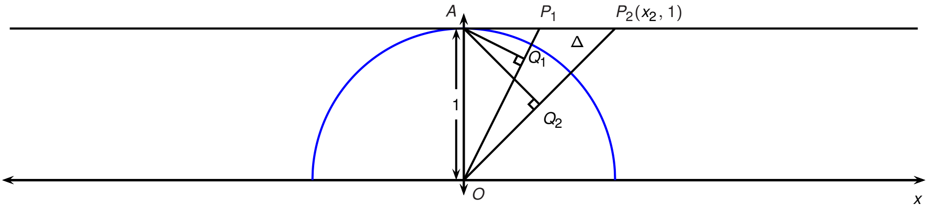

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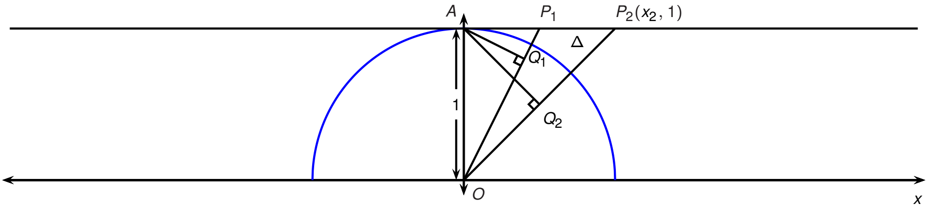

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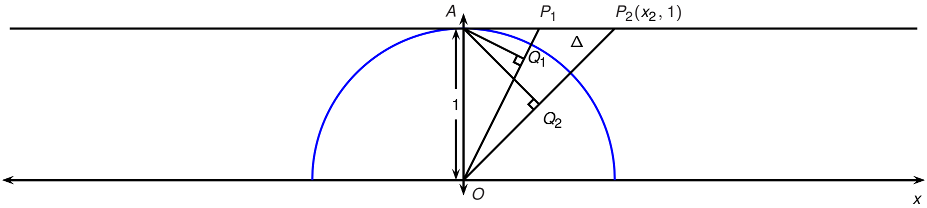

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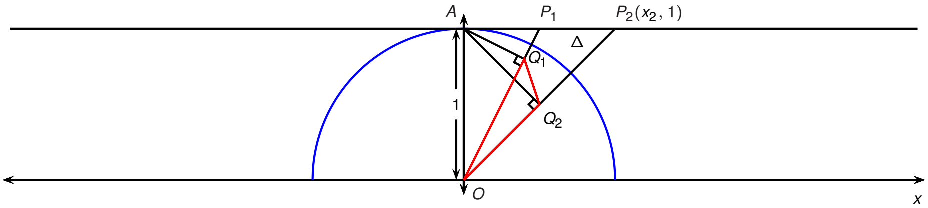

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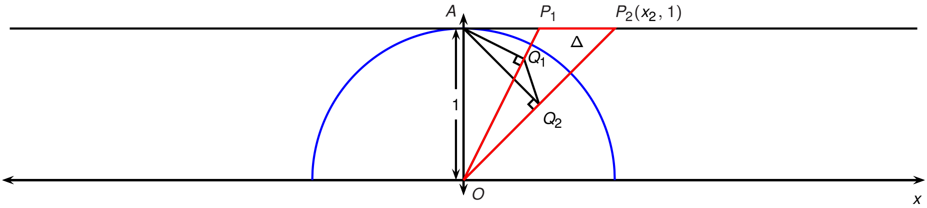

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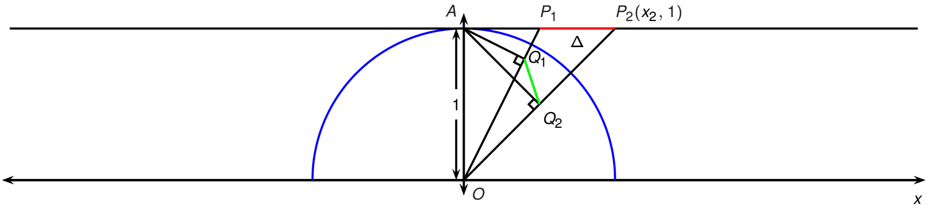

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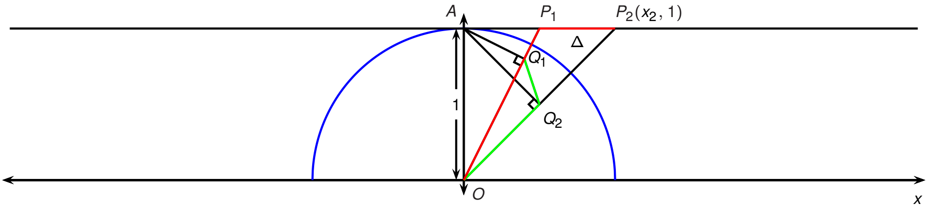

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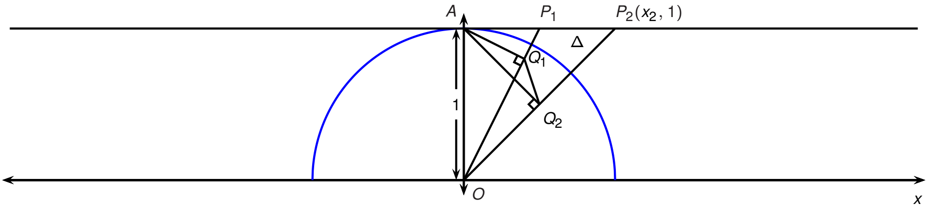

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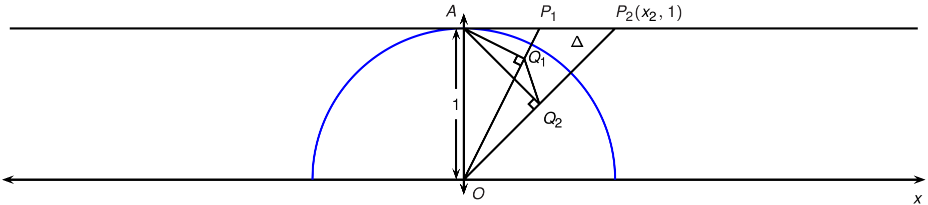

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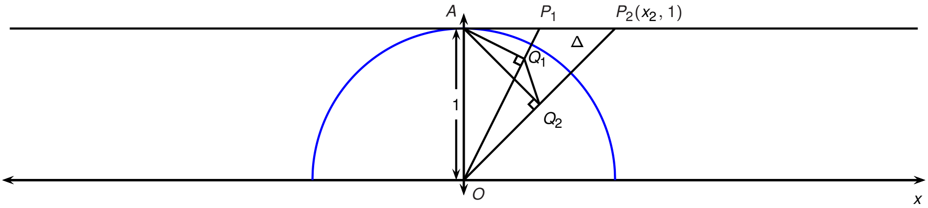

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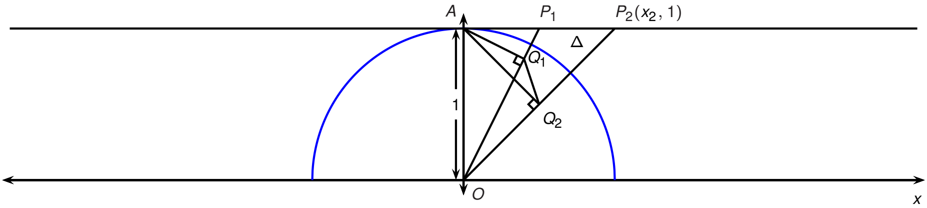
$$|Q_1Q_2| = \frac{|P_1P_2||OQ_2|}{|OP_1|} = \left(\frac{|OP_2|}{|OP_1|}\right)\frac{|OQ_2|}{|OP_2|}|P_1P_2|$$



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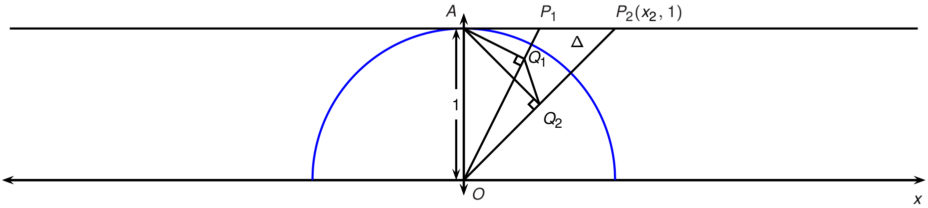

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$$\frac{|OQ_2|}{|OP_2|} = \frac{|OQ_2||OP_2|}{|OP_2|^2}$$

Similarly conclude $|OQ_1||OP_1| = |OA|^2 = 1 = |OQ_2||OP_2|$. Therefore $\frac{|OQ_1|}{|OP_2|} = \frac{|OQ_2|}{|OP_1|}$ and so $\triangle OQ_2Q_1$ is similar to $\triangle OP_1P_2$. Therefore $\frac{|Q_1Q_2|}{|P_1P_2|} = \frac{|OQ_2|}{|OP_1|}$ and so

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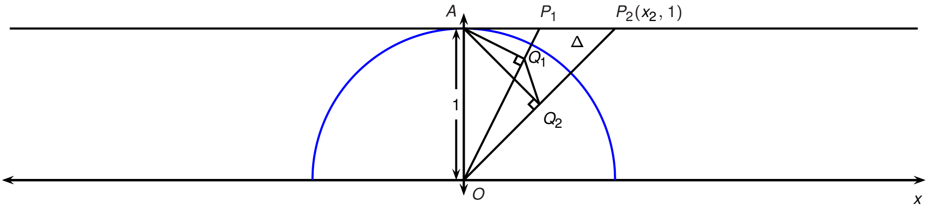

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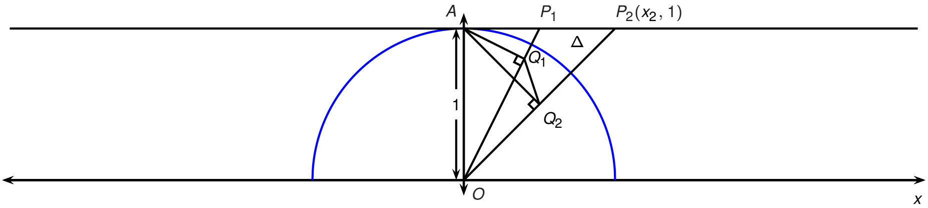

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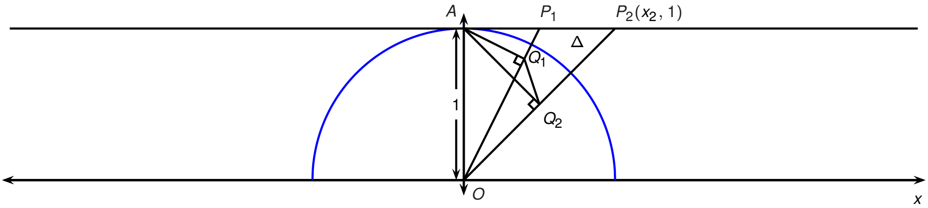

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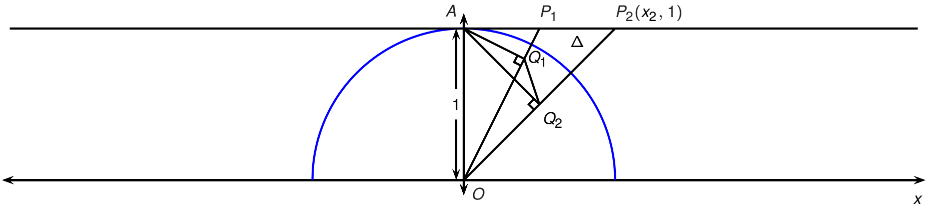

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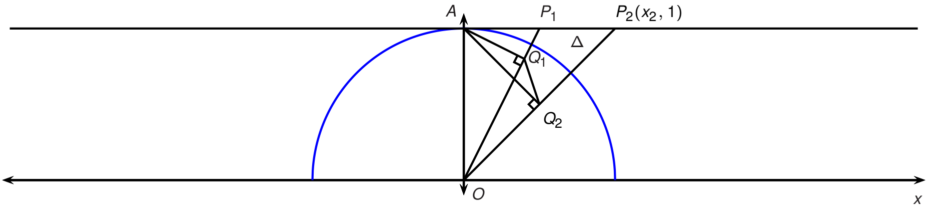

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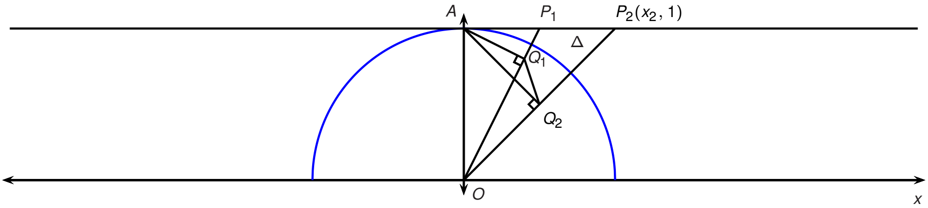

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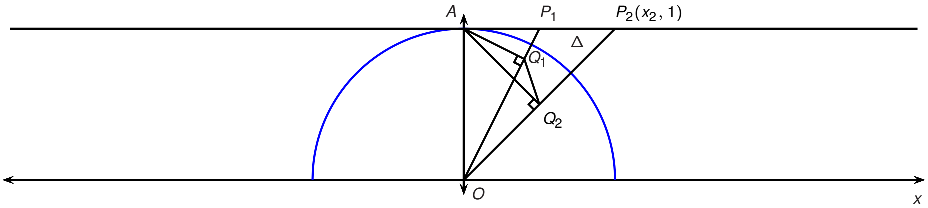
If we let $P_2 \to P_1$



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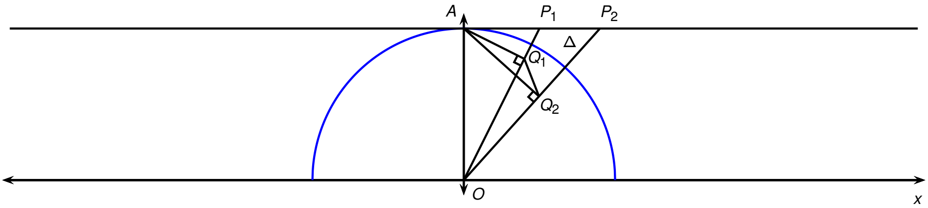
If we let $P_2 \to P_1$, i.e., $\Delta \to 0$,



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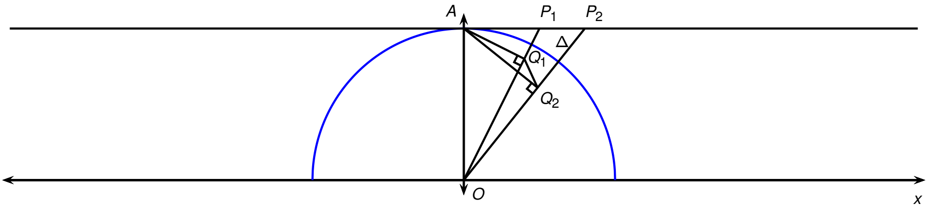

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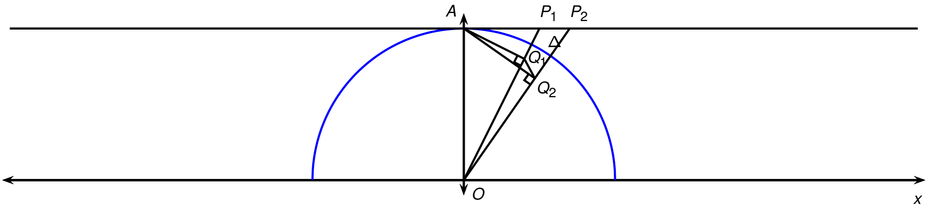

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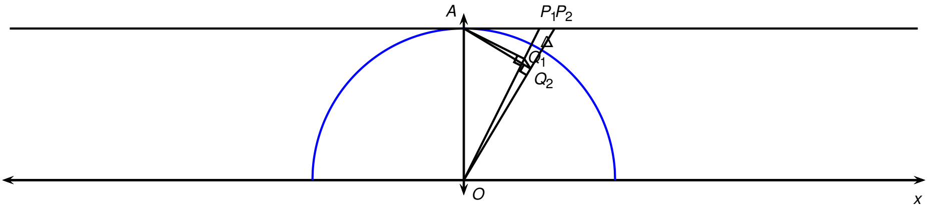

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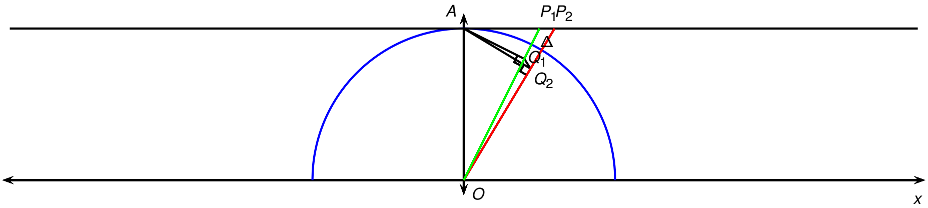

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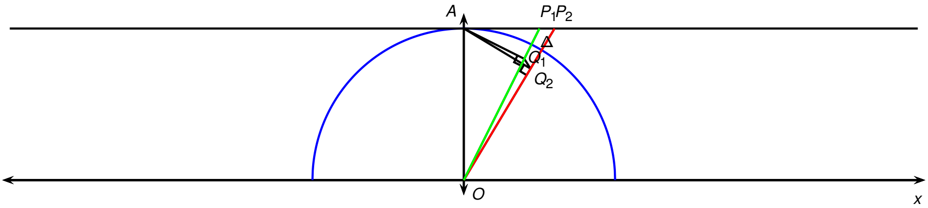
If we let $P_2 \to P_1$, i.e., $\Delta \to 0$, we get $\frac{|OP_2|}{|OP_1|} \to 1$.



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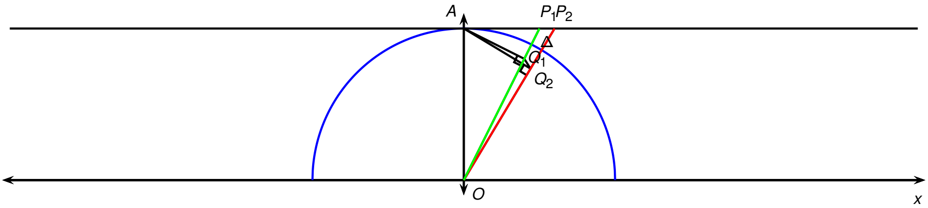

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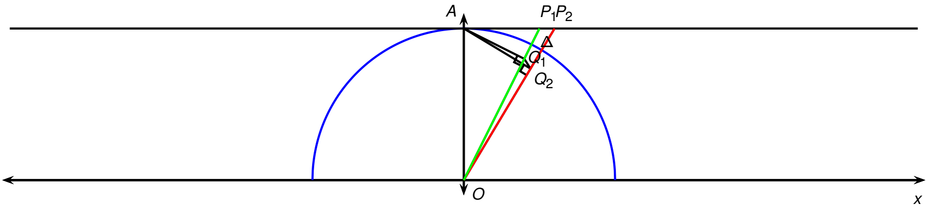

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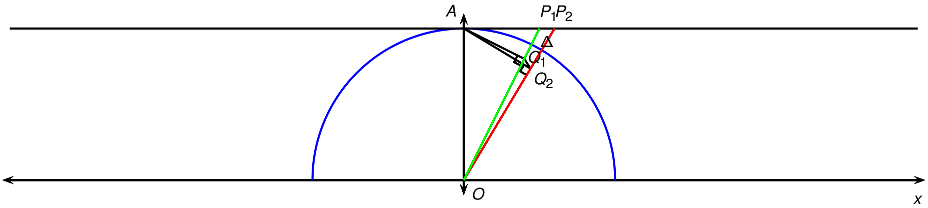
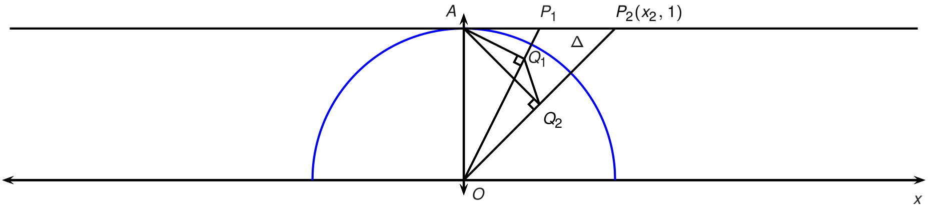


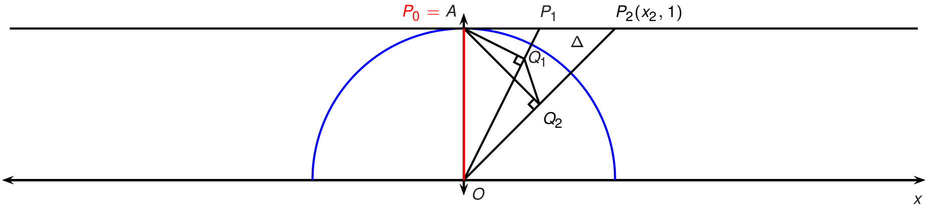
$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$.

If we let $P_2 \to P_1$, i.e., $\Delta \to 0$, we get $\frac{|OP_2|}{|OP_1|} \to 1$. In strict mathematical language: for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $\Delta < \delta$ we have that $1 > \frac{|OP_2|}{|OP_1|} > 1 - \varepsilon$. Furthermore, the choice of δ can be made independent of the value of x_2: to prove that one analyzes the expression $\frac{|OP_2|}{|OP_1|} = \sqrt{\frac{1+x_2^2}{1+(x_2-\Delta)^2}}$. We leave the tedious but otherwise easy details to the interested student.



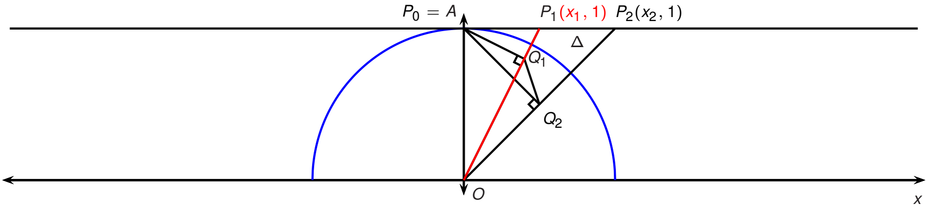
$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer.



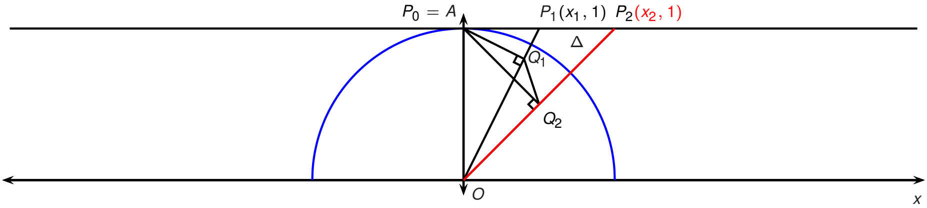
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer. Let $P_0 = (0, 1)$, $P_1 = (\Delta, 1)$, $P_2 = (2\Delta, 1), \ldots, P_n = (n\Delta, 1)$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer. Let $P_0 = (0, 1)$, $P_1 = (\Delta, 1)$, $P_2 = (2\Delta, 1), \ldots, P_n = (n\Delta, 1)$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.
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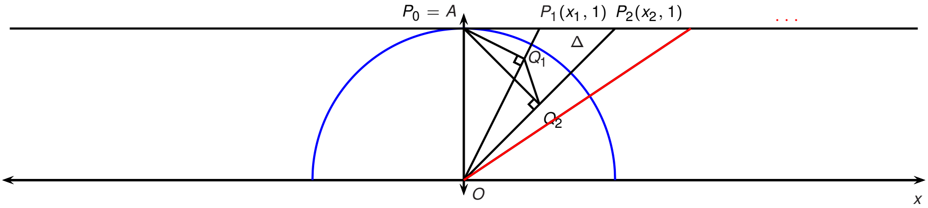
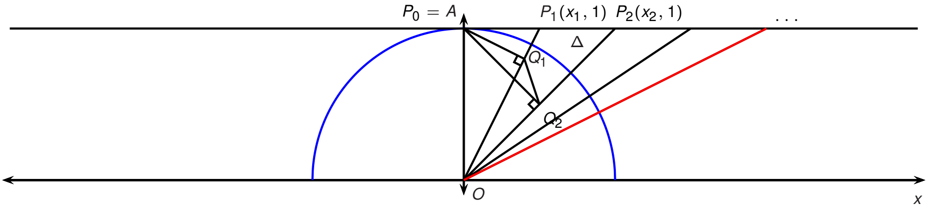
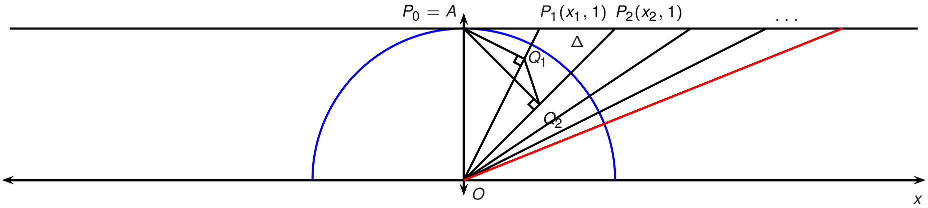


$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer. Let $P_0 = (0, 1)$, $P_1 = (\Delta, 1)$, $P_2 = (2\Delta, 1), \ldots, P_n = (n\Delta, 1)$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.
Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer. Let $P_0 = (0, 1)$, $P_1 = (\Delta, 1)$, $P_2 = (2\Delta, 1), \dots, P_n = (n\Delta, 1)$



$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

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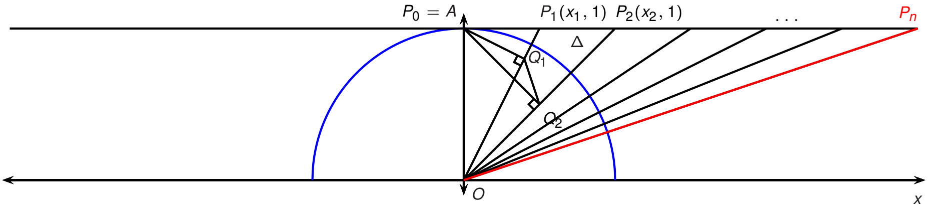

$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

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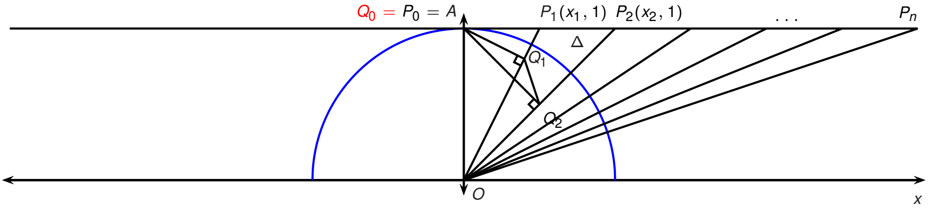

$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.
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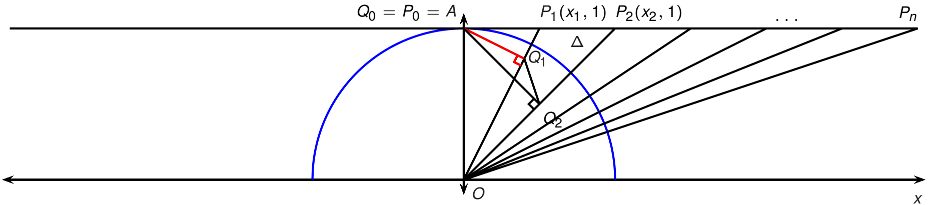

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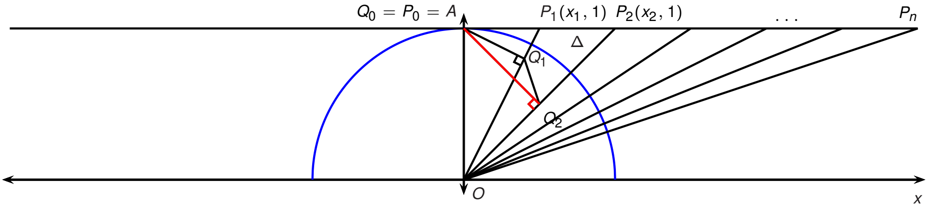

$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

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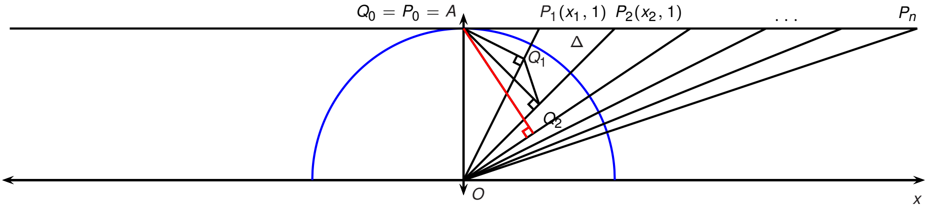

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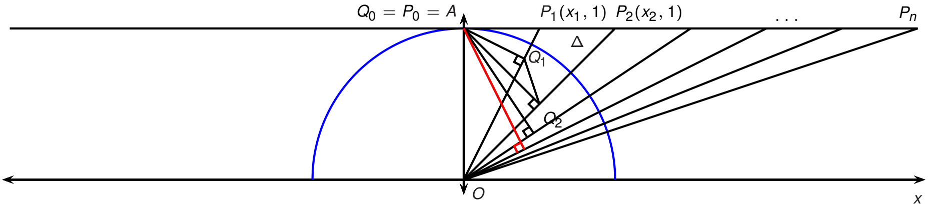

$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

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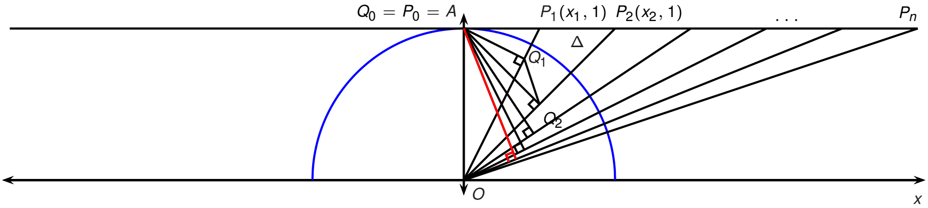

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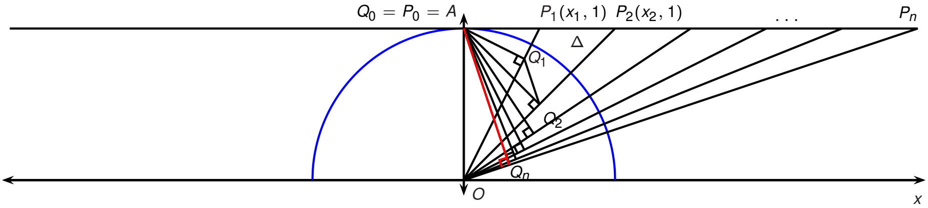

$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

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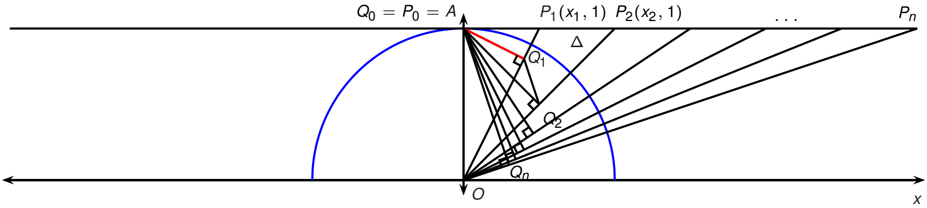
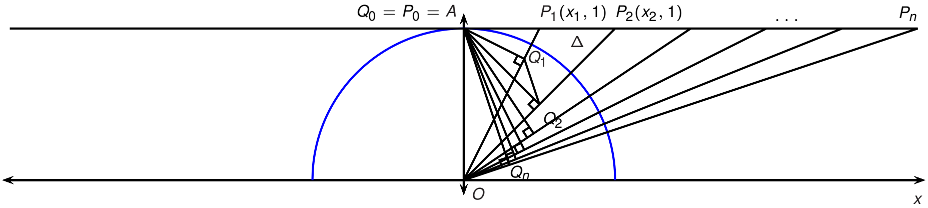


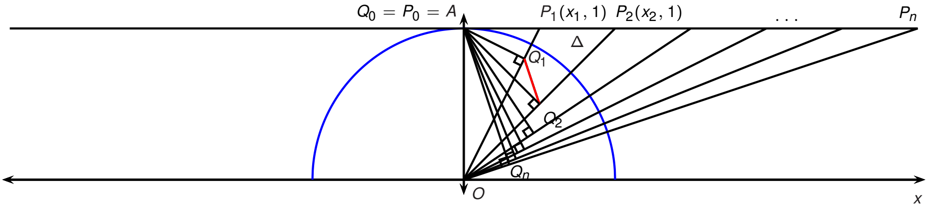
$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\frac{\Delta}{1+x_1^2} \quad < \quad |Q_0 Q_1| \quad < \quad (1+\varepsilon)\frac{\Delta}{1+x_1^2}$$



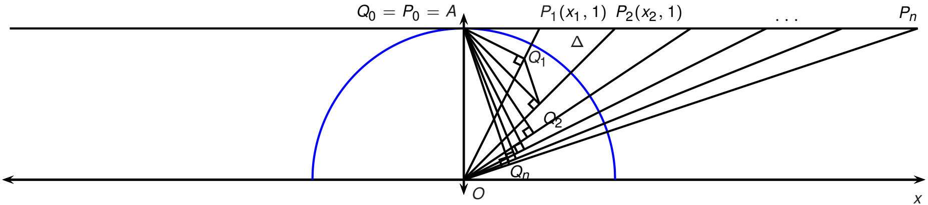
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\frac{\Delta}{1+x_1^2} \quad < \quad |Q_0Q_1| \quad < \quad (1+\varepsilon)\frac{\Delta}{1+x_1^2}$$



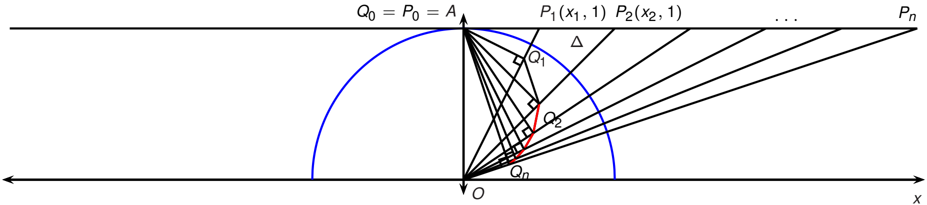
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\frac{\Delta}{1+x_1^2} < |Q_0Q_1| < (1+\varepsilon)\frac{\Delta}{1+x_1^2}$$
$$\frac{\Delta}{1+x_2^2} < |Q_1Q_2| < (1+\varepsilon)\frac{\Delta}{1+x_2^2}$$



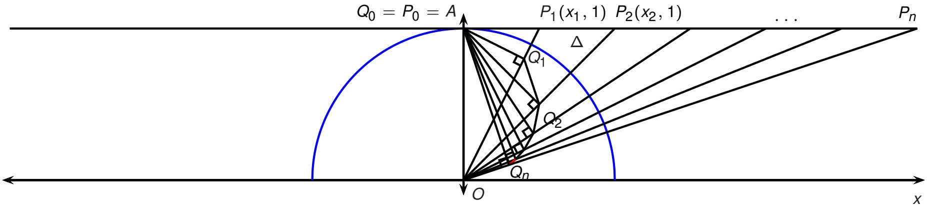
$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\frac{\Delta}{1+x_1^2} < |Q_0 Q_1| < (1+\varepsilon)\frac{\Delta}{1+x_1^2}$$
$$\frac{\Delta}{1+x_2^2} < |Q_1 Q_2| < (1+\varepsilon)\frac{\Delta}{1+x_2^2}$$



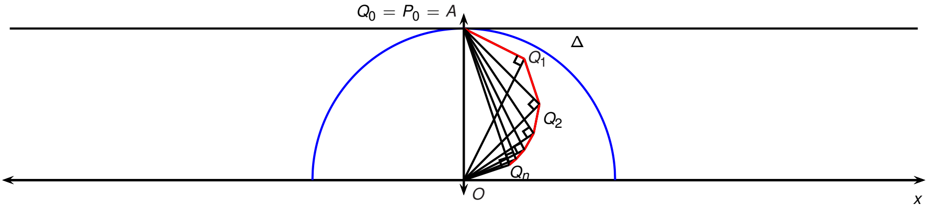
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\frac{\Delta}{1+x_1^2} < |Q_0Q_1| < (1+\varepsilon)\frac{\Delta}{1+x_1^2}$$
$$\frac{\Delta}{1+x_2^2} < |Q_1Q_2| < (1+\varepsilon)\frac{\Delta}{1+x_2^2}$$
$$\vdots$$



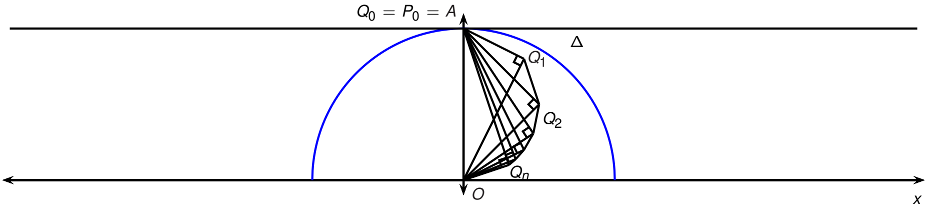
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2}
\end{array}$$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

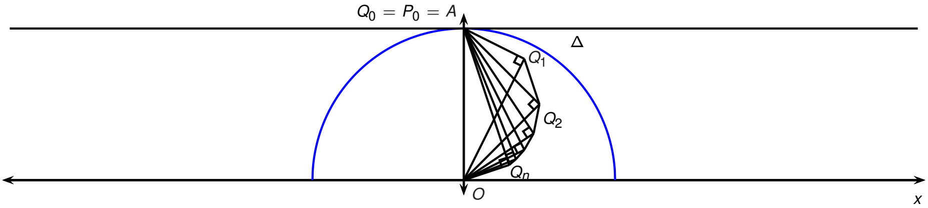
$$
\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \color{red}{\sum_{i=1}^n |Q_{i-1}Q_i|} & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2}
\end{array}
$$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$
\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^n |Q_{i-1}Q_i| & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^N \frac{dx}{1+x^2} & < & \lim_{\Delta} \sum |Q_{i-1}Q_i| & < & (1+\varepsilon)\int_0^N \frac{dx}{1+x^2}
\end{array}
$$

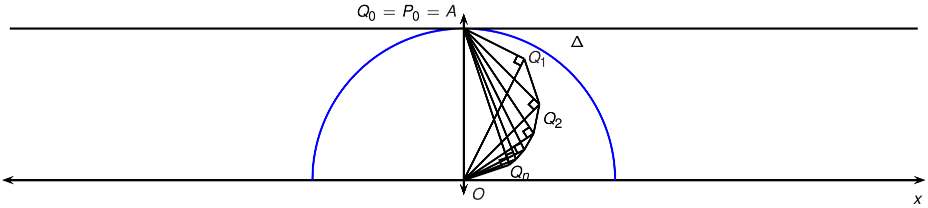
Let $\Delta \to 0$.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^n |Q_{i-1}Q_i| & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^N \frac{dx}{1+x^2} & < & \lim\limits_{\Delta} \sum |Q_{i-1}Q_i| & < & (1+\varepsilon)\int_0^N \frac{dx}{1+x^2}
\end{array}$$

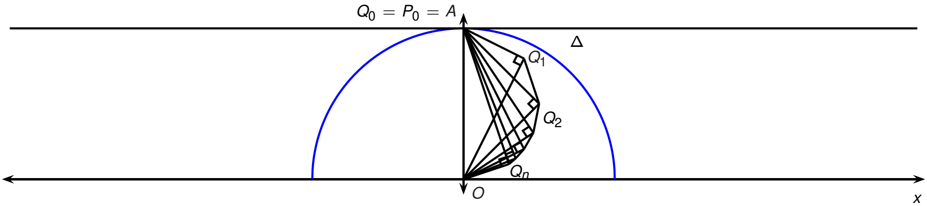
Let $\Delta \to 0$. Next take $N \to \infty$.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$
\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^n |Q_{i-1}Q_i| & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^\infty \frac{dx}{1+x^2} & < & \lim\limits_{\Delta, N} \sum |Q_{i-1}Q_i| & < & (1+\varepsilon)\int_0^\infty \frac{dx}{1+x^2}
\end{array}
$$

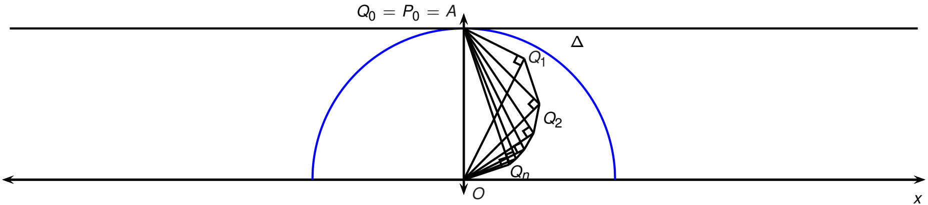
Let $\Delta \to 0$. Next take $N \to \infty$.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$
\begin{array}{rcccl}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^n |Q_{i-1}Q_i| & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^\infty \frac{dx}{1+x^2} & < & \lim\limits_{\Delta, N} \sum |Q_{i-1}Q_i| & < & (1+\varepsilon)\int_0^\infty \frac{dx}{1+x^2}
\end{array}
$$

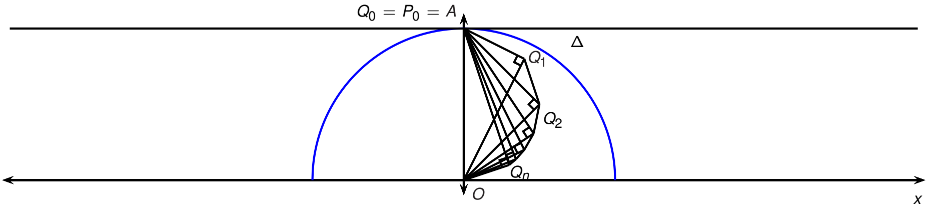
Let $\Delta \to 0$. Next take $N \to \infty$. Finally take $\varepsilon \to 0$



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

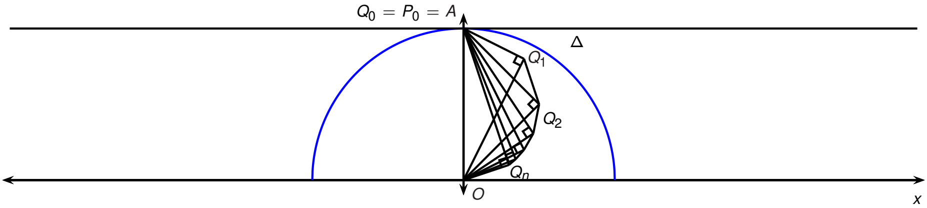
$$\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1}Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^n \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^n |Q_{i-1}Q_i| & < & (1+\varepsilon)\sum_{i=1}^n \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^\infty \frac{dx}{1+x^2} & = & \lim\limits_{\Delta,N,\varepsilon} \sum |Q_{i-1}Q_i| & = & \int_0^\infty \frac{dx}{1+x^2}
\end{array}$$

Let $\Delta \to 0$. Next take $N \to \infty$. Finally take $\varepsilon \to 0$, use squeeze thm.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\Delta,N,\varepsilon} \sum |Q_{i-1}Q_i|$$



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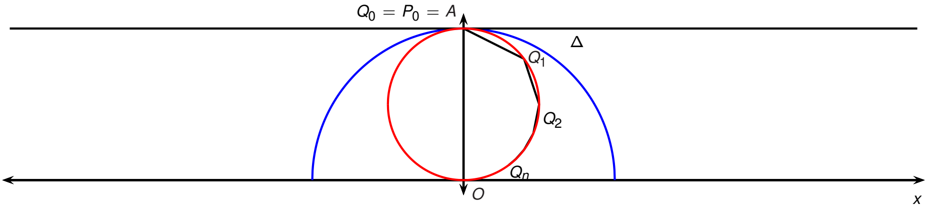
$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1} Q_i|$$

The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$.



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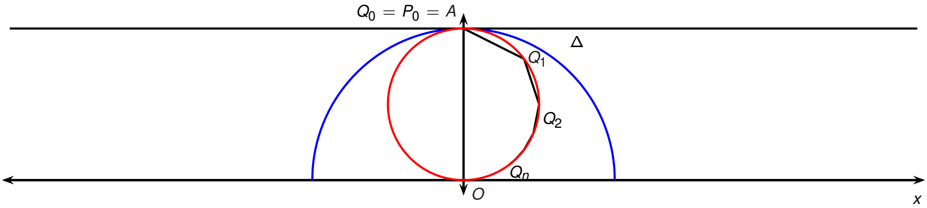
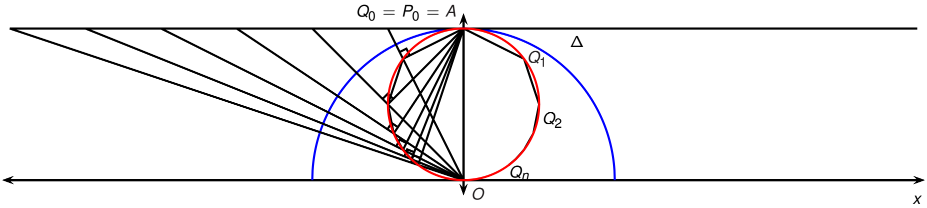
The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \ldots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1}Q_i|$$

The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \ldots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. Therefore $\sum |Q_{i-1}Q_i|$ approximates half of the circumference of the circle C.



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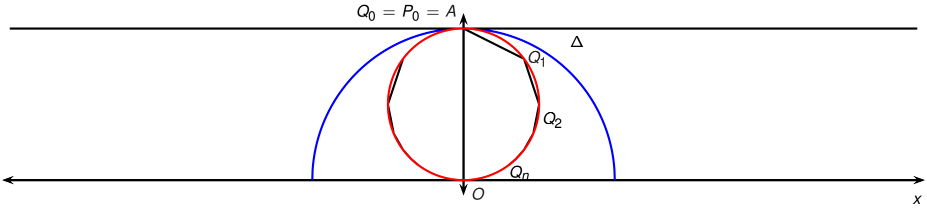

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$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1} Q_i|$$

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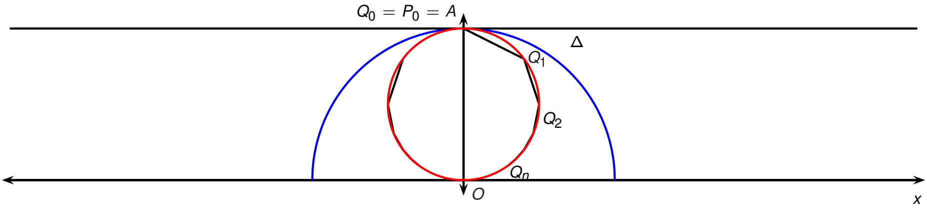
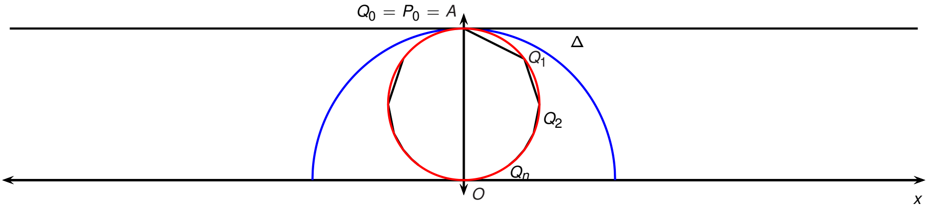


$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$.

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1}Q_i|$$

The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \ldots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. Therefore $\sum |Q_{i-1}Q_i|$ approximates half of the circumference of the circle C. By symmetry,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \text{circumference of } C$$



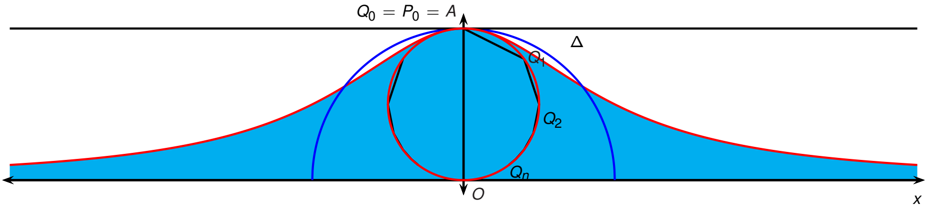
$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1} Q_i|$$

The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \ldots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. Therefore $\sum |Q_{i-1} Q_i|$ approximates half of the circumference of the circle C. By symmetry,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \text{circumference of } C = 2\pi \left(\frac{1}{2}\right) = \pi,$$

as desired.



$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|} \frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\Delta, N, \varepsilon} \sum |Q_{i-1} Q_i|$

The points $Q_1, Q_2, \ldots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \ldots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. Therefore $\sum |Q_{i-1} Q_i|$ approximates half of the circumference of the circle C. By symmetry,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \text{ circumference of } C = 2\pi \left(\frac{1}{2}\right) = \pi,$$

as desired.