Math 141
Lecture 10

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Outline

1 Sequences

Example (A finite sequence)

$$1, 2, 3, 4, 5$$

is an example of a finite sequence. So is

$$-1, -2, -3, \ldots, -10000.$$

Definition (Finite sequence, Infinite sequence)

A finite sequence is a sequence that ends. It is possible to write down all the terms in a finite sequence. A sequence that is not finite is called an infinite sequence.

Example (A finite sequence)

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A finite sequence is a sequence that ends. It is possible to write down all the terms in a finite sequence. A sequence that is not finite is called an infinite sequence.

Example (An infinite sequence)

$$2, 4, 6, 8, \ldots$$

is an example of an infinite sequence.

Example (Sequence notation)

Consider the sequence

$$2, 4, 6, 8, \ldots.$$

We can express this sequence more compactly using the notation

$$a_n = 2n,$$

where a_n denotes the nth term.

$$\text{So} \quad \begin{aligned} a_1 &= 2 \cdot 1 = 2 \\ a_2 &= 2 \cdot 2 = 4 \\ a_3 &= 2 \cdot 3 = 6 \\ a_4 &= 2 \cdot 4 = 8 \\ &\vdots \end{aligned}$$

Example

The sequence

$$-1, 1, -1, 1, -1, 1, \dots$$

can be written $b_n = (-1)^n$.

Example

The sequence

$$1, 2, 4, 8, 16, \dots$$

can be written $c_n = 2^{n-1}$.

Example

The sequence

$$\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$$

can be written $d_n = -\big(-\frac{1}{2}\big)^n$.

Example (Given a formula, find the terms)

Find the first five terms of each of the following sequences.

1. $a_n = 3 \cdot 2^{-n}$
2. $b_n = 1$
3. $c_n = -3(n-1) + 5$
4. $d_n = n^2 + 1$

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$$1, 1, 1, 1, 1, \ldots$$

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2. $b_n = 1$

$$1, 1, 1, 1, 1, \ldots$$

3. $c_n = -3(n-1) + 5$

$$5, 2, -1, -4, -7, \ldots$$

4. $d_n = n^2 + 1$

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2. $b_n = 1$

$$1, 1, 1, 1, 1, \ldots$$

3. $c_n = -3(n-1) + 5$

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2. $b_n = 1$

$$1, 1, 1, 1, 1, \ldots$$

3. $c_n = -3(n-1) + 5$

$$5, 2, -1, -4, -7, \ldots$$

4. $d_n = n^2 + 1$

$$2, 5, 10, 17, 26, \ldots$$

Example (Given the terms, find a formula)

Find a formula for the *n*th term of each of the following sequences.

1. $a_n =$

$$2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \ldots$$

2. $b_n =$

$$-1, 4, -9, 16, -25, \ldots$$

3. $c_n =$

$$-1, 5, 11, 17, 23, \ldots$$

Example (Given the terms, find a formula)

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1. $a_n =$
$$2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \ldots$$
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Find a formula for the *n*th term of each of the following sequences.

1. $a_n = 2 \cdot \left(\frac{1}{4}\right)^{n-1}$

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2. $b_n =$

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2. $b_n = (-1)^n n^2$

$$-1, 4, -9, 16, -25, \ldots$$

3. $c_n = -1 + 6(n-1)$

$$-1, 5, 11, 17, 23, \ldots$$

Definition (Arithmetic sequence)

An arithmetic sequence is one in which successive terms differ by a constant number. This constant is called the difference of the arithmetic sequence.

Example (Which are arithmetic?)

1,	2,	3,	4,	5,	...	is arithmetic with difference 1.
23,	16,	9,	2,	-5,	...	is arithmetic with difference -7.
8,	9,	12,	17,	24,	...	is not arithmetic.
						($9-8=1$ but $12-9=3$.)

Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$				
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
$2, 2, 2, 2, \ldots$				

Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
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Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$	no	—		
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
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Example (Which are arithmetic?)

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Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$	no	—	1	
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
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Example (Which are arithmetic?)

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Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$	no	—	1	$(-1)^{n+1}$
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
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$2, 2, 2, 2, \ldots$	yes			

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \dots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \dots$	yes			

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0		

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0		

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0	2	

Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$	no	—	1	$(-1)^{n+1}$
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0	2	

Example (Which are arithmetic?)

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0	2	2

Example (Which are arithmetic?)

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$1, -1, 1, -1, \ldots$	no	—	1	$(-1)^{n+1}$
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
$2, 2, 2, 2, \ldots$	yes	0	2	$2 + 0(n-1)$

If an arithmetic sequence has difference d, then the nth term has formula

$$a_n = a_1 + d(n-1),$$

where a_1 is the first term.

Definition (Geometric sequence)

A geometric sequence is one in which each term is obtained by multiplying the previous one by the same constant. This constant is called the ratio of the geometric sequence.

Example (Which are geometric?)

$2,$	$4,$	$8,$	$16,$	$32,$	$\dots$	is geometric with ratio 2.
$1,$	$-3,$	$9,$	$-27,$	$81,$	$\dots$	is geometric with ratio -3.
$-42,$	$-14,$	$-21,$	$31,$	$-22,$	$\dots$	is not geometric. ($\frac{-14}{-42} = \frac{1}{3}$ but $\frac{-21}{-14} = \frac{3}{2}$.)

Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$					
$7, 3, -1, -5, \ldots$					
$4, 4, 4, 4, \ldots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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$1, 1, 2, 2, 3, 3, \ldots$					

Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$	geometric	—			
$7, 3, -1, -5, \ldots$					
$4, 4, 4, 4, \ldots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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$7, 3, -1, -5, \ldots$					
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$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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$7, 3, -1, -5, \ldots$					
$4, 4, 4, 4, \ldots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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$7, 3, -1, -5, \ldots$					
$4, 4, 4, 4, \ldots$					
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$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$	geometric	—	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^n$
$7, 3, -1, -5, \ldots$					
$4, 4, 4, 4, \ldots$					
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$4, 4, 4, 4, \ldots$					
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$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$	geometric	—	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^n$
$7, 3, -1, -5, \ldots$	arithmetic		—		
$4, 4, 4, 4, \ldots$					
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Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$	geometric	—	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^n$
$7, 3, -1, -5, \ldots$	arithmetic	-4	—		
$4, 4, 4, 4, \ldots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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$7, 3, -1, -5, \ldots$	arithmetic	-4	—	7	
$4, 4, 4, 4, \ldots$					
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$4, 4, 4, 4, \ldots$	both	0			
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$					
$1, 1, 2, 2, 3, 3, \ldots$					

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Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$	geometric	—	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^n$
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$4, 4, 4, 4, \ldots$	both	0	1	4	
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$4, 4, 4, 4, \ldots$	both	0	1	4	4
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$4, 4, 4, 4, \ldots$	both	0	1	4	4
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
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$4, 4, 4, 4, \ldots$	both	0	1	4	4
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
$1, 1, 2, 2, 3, 3, \ldots$	neither	—	—		

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$4, 4, 4, 4, \dots$	both	0	1	4	4
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
$1, 1, 2, 2, 3, 3, \dots$	neither	—	—	1	

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$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
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$4, 4, 4, 4, \ldots$	both	0	1	4	4
$\pi, -\pi^2, \pi^3, -\pi^4, \ldots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
$1, 1, 2, 2, 3, 3, \ldots$	neither	—	—	1	$\lceil \frac{n}{2} \rceil$

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$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$	geometric	—	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{2}{3}\right)^n = \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$
$7, 3, -1, -5, \dots$	arithmetic	-4	—	7	$7 - 4(n-1)$
$4, 4, 4, 4, \dots$	both	0	1	4	$4 = 4(1)^{n-1}$
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
$1, 1, 2, 2, 3, 3, \dots$	neither	—	—	1	$\lceil \frac{n}{2} \rceil$

If a geometric sequence has ratio r, then the nth term has formula

$$a_n = a_1 r^{n-1}.$$

where a_1 is the first term.

Sequences

Definition (Sequence)

A sequence is a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is called the second term, and in general a_n is the nth term.
We will always deal with infinite sequences, in which each term a_n has a successor a_{n+1}.

Notation:
The sequence $\{a_1, a_2, a_3, \ldots\}$ can also be written

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Example (A sequence)

$$1, 2, 3, 4, 5, \ldots$$

is an example of a sequence.

Example (A sequence)

$$1, 2, 3, 4, 5, \dots$$

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Definition (Sequence, Terms)

A sequence is a list of numbers written in a definite order. The individual numbers in the sequence are called the terms of the sequence.

Example (A sequence)

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is an example of a sequence.

Definition (Sequence, Terms)

A sequence is a list of numbers written in a definite order. The individual numbers in the sequence are called the terms of the sequence.

Example (More sequences)

$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad \ldots$ is a sequence.

$-1, \quad 1, \quad -1, \quad 1, \quad -1, \quad 1, \quad \ldots$ is a sequence.

Example (A sequence)

$$1, 2, 3, 4, 5, \ldots$$

is an example of a sequence.

Definition (Sequence, Terms)

A sequence is a list of numbers written in a definite order. The individual numbers in the sequence are called the terms of the sequence.

Example (More sequences)

$$\begin{array}{rrrrrrll} 1, & 2, & 4, & 8, & 16, & 32, & \ldots & \text{is a sequence.} \\ -1, & 1, & -1, & 1, & -1, & 1, & \ldots & \text{is a sequence.} \\ 1, & -1, & 1, & -1, & 1, & -1, & \ldots & \text{is a different sequence.} \end{array}$$

Some sequences can by defined by giving a formula for the nth term a_n. This example expresses four different sequences in three different ways: first, by using the preceding notation; second, by giving a formula; and third, by writing out the terms of the sequence.

Example

$\left\{\frac{n}{n+1}\right\}$	$a_n = \frac{n}{n+1}$	$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$
$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$	$a_n = \frac{(-1)^n(n+1)}{3^n}$	$\left\{\frac{-2}{3}, \frac{3}{9}, \frac{-4}{27}, \frac{5}{81}, \ldots\right\}$
$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3}, n \geq 3$	$\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots\right\}$
$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$	$a_n = \cos\frac{n\pi}{6}, n \geq 0$	$\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots\right\}$

Example

Find a formula for the general term a_n of the sequence

$$\left\{0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right\}$$

Example

Find a formula for the general term a_n of the sequence

$$\left\{0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right\}$$

$$a_1 = 0,\ a_2 = \frac{1}{4},\ a_3 = -\frac{2}{8},\ a_4 = \frac{3}{16},\ a_5 = -\frac{4}{32},\ a_6 = \frac{5}{64},$$

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- The numerators start at 0 and go up by one with each term.

Example

Find a formula for the general term a_n of the sequence

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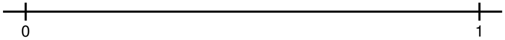
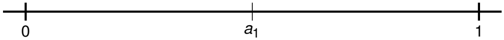
The Fibonacci sequence can be described by a formula, but not a simple one $\left(a_n = \frac{\sqrt{5}}{5} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \right)$.



- The sequence $\left\{\frac{n}{n+1}\right\}$ can be plotted on a number line or using Cartesian coordinates.



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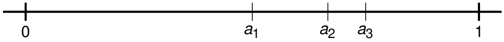

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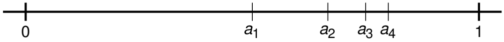

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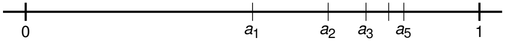

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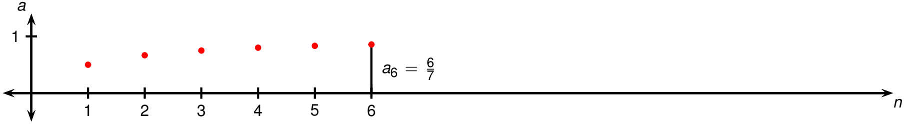

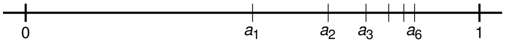

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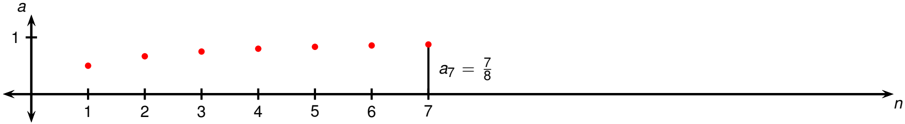

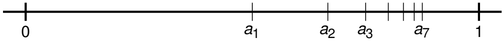

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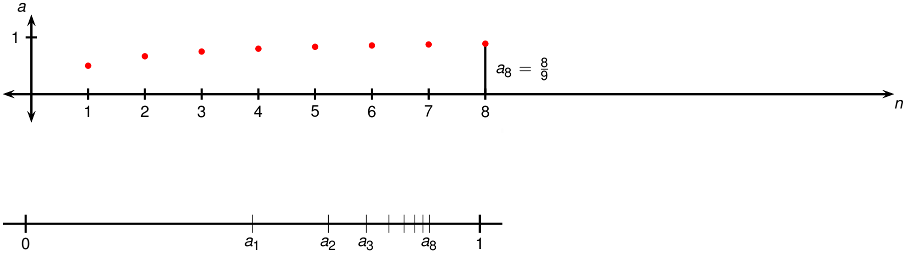

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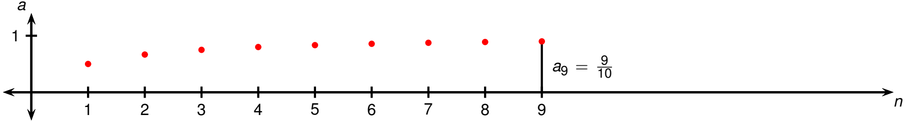

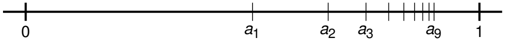

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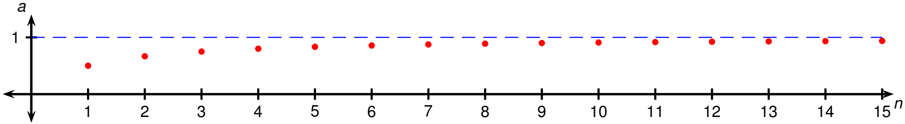

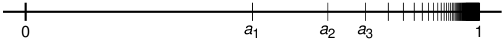

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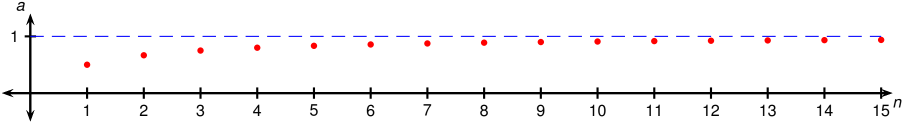

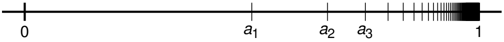

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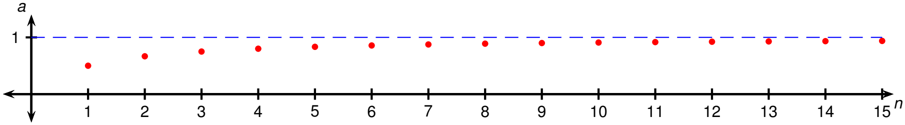
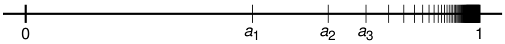
- From the pictures, the terms in the sequence appear to approach 1 as n gets larger.





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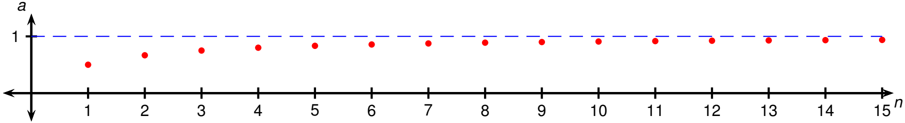

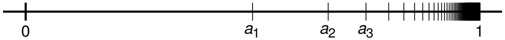

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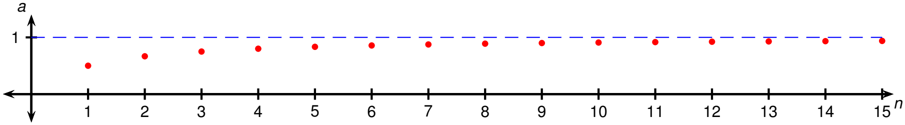

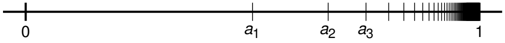

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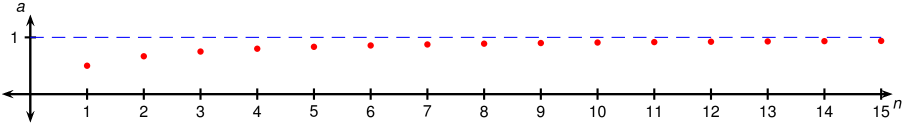

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- This can be made arbitrarily small by choosing n large enough.
- We express this by writing $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Definition (Limit of a Sequence)

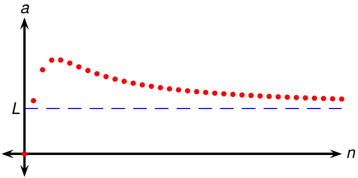
A sequence $\{a_n\}$ has the limit L, and we write

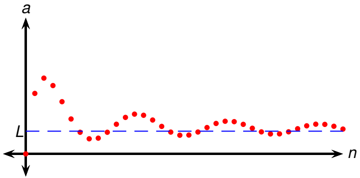
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make a_n as close to L as we like by taking n large enough.

Definition (Convergent)

A sequence that has a limit is called convergent. A sequence that has no limit is called divergent.

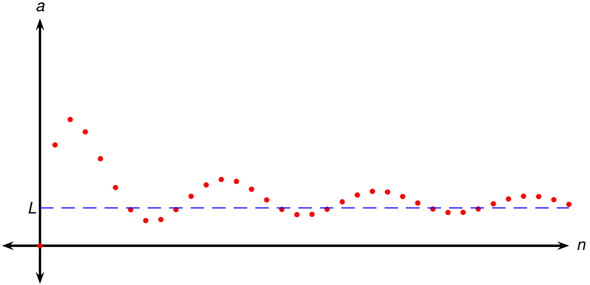




If you compare the definition of the limit of a sequence with the definition of the infinite limit of a function, you'll see that the only difference between

$$\lim_{n\to\infty} a_n = L \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = L$$

is that n is required to be an integer.



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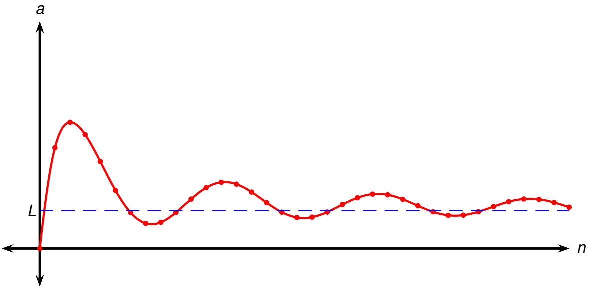

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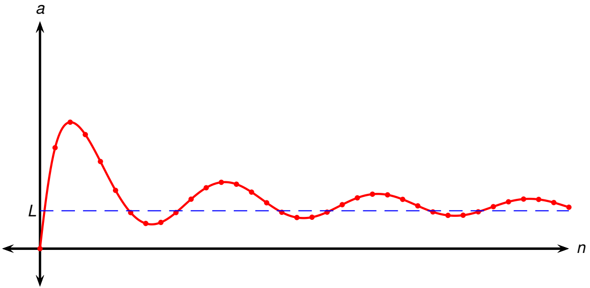

Theorem

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *for all integers n, then* $\lim_{n\to\infty} a_n = L$.

Example

Find $\lim_{n\to\infty} \frac{n}{n+1}$.

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Divide numerator and denominator by the highest power of n, and use the limit laws:

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&= \frac{1}{1+0} \\
&= 1
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You can probably guess what $\lim_{n\to\infty} a_n = -\infty$ means.

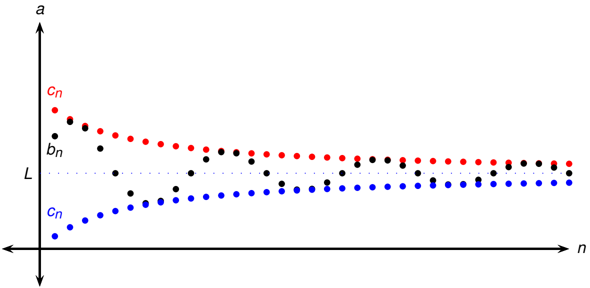
The Limit Laws from section 2.3 also hold for sequences:
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

The Squeeze Theorem also works for sequences:

Theorem (The Squeeze Theorem for Sequences)

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.



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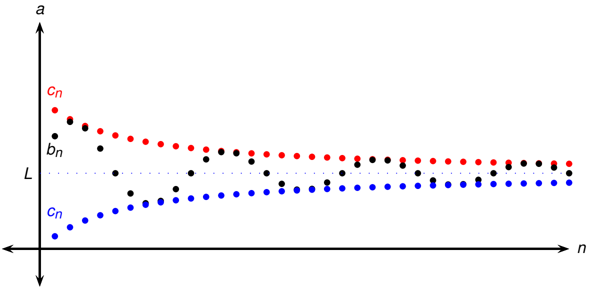

Corollary

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

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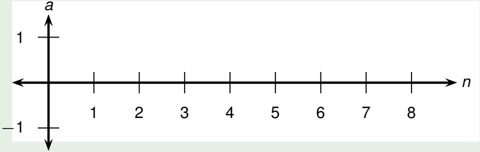
$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} f(x) = 0$$

Example

Is the sequence $a_n = (-1)^n$ convergent or divergent?



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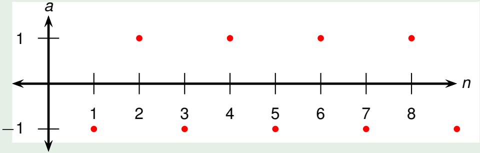

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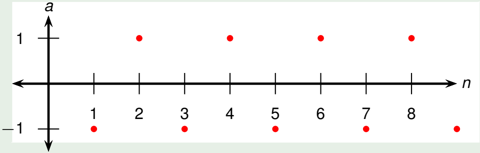

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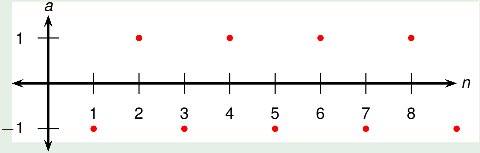

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- $\{a_n\}$ is divergent.

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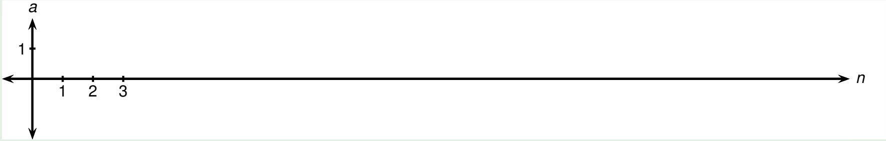

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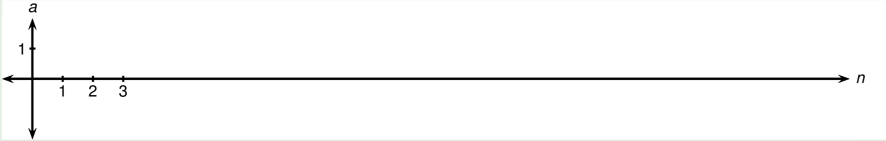

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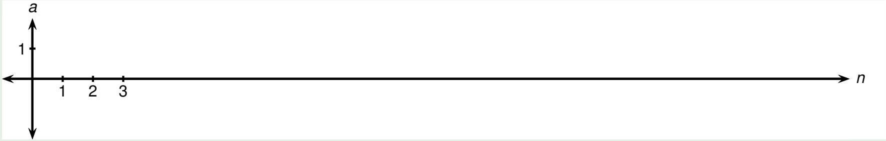

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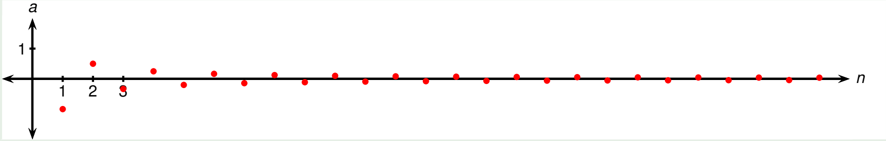

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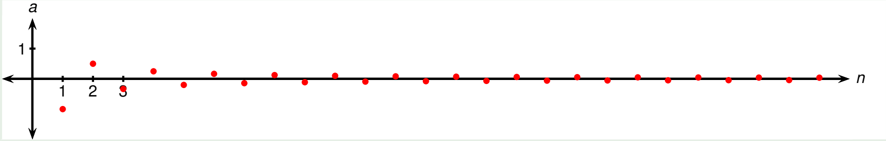
Therefore $\left\{ \frac{(-1)^n}{n} \right\}$ is convergent.



Theorem

If $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example

Find $\lim_{n\to\infty} \sin(\pi/n)$.

$$\lim_{n\to\infty} \sin(\pi/n)$$

Find $\lim_{n\to\infty} \cos(\pi/n)$.

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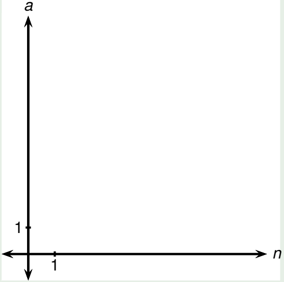
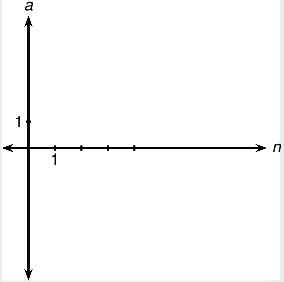
- Since $\frac{1}{n} \to 0$ as $n \to \infty$, by the Squeeze Theorem $a_n \to 0$ as $n \to \infty$.

Example

For what values of r is the sequence $\{r^n\}$ convergent?





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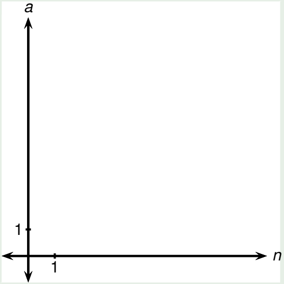

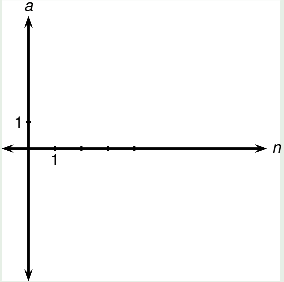

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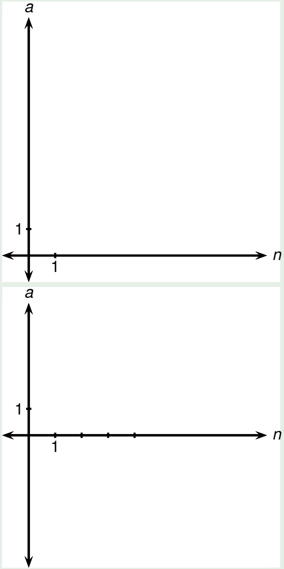

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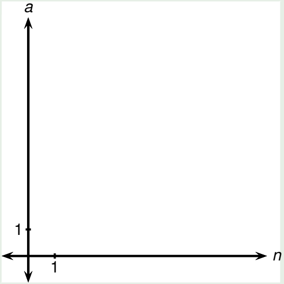

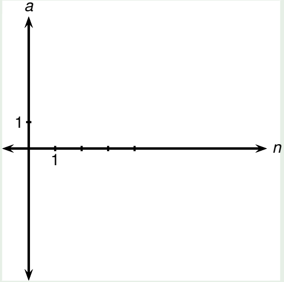

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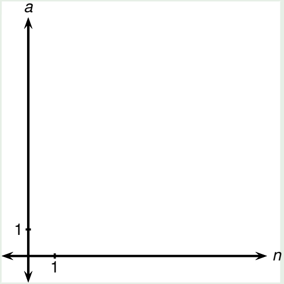

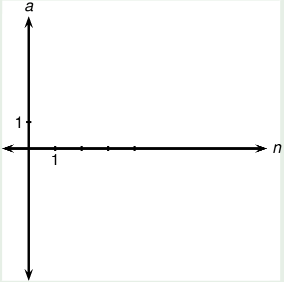

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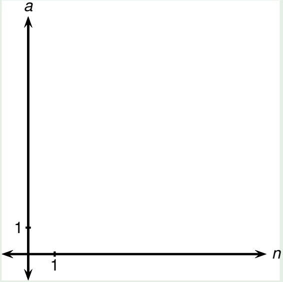

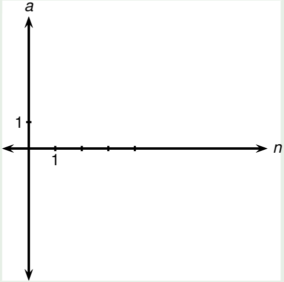

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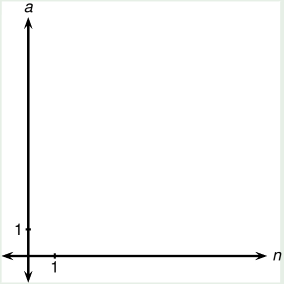

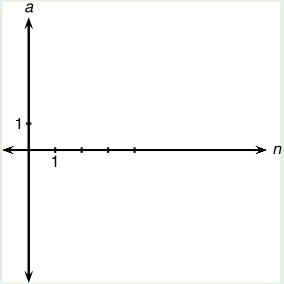

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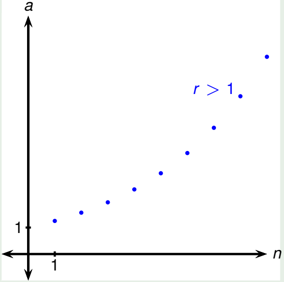

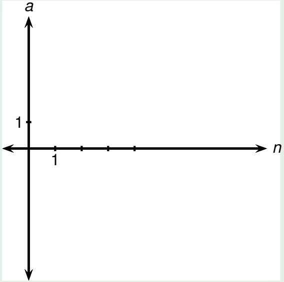

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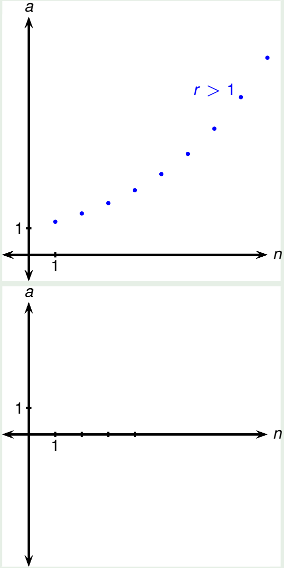

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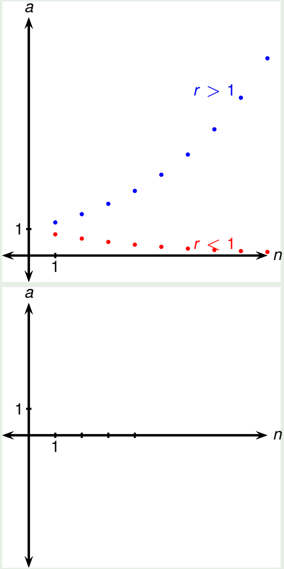

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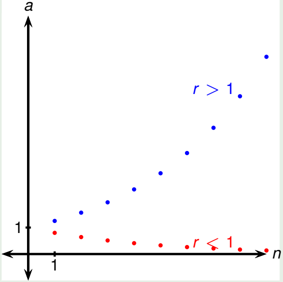

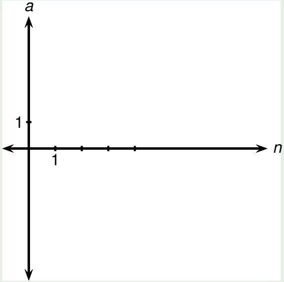

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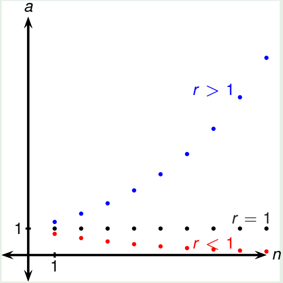

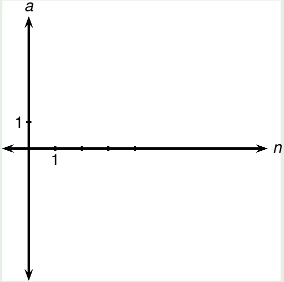

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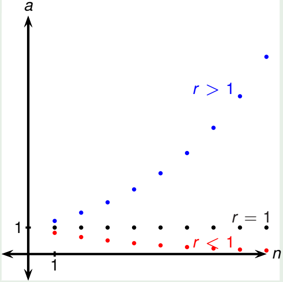

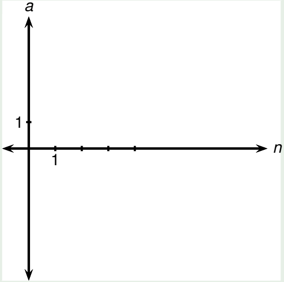

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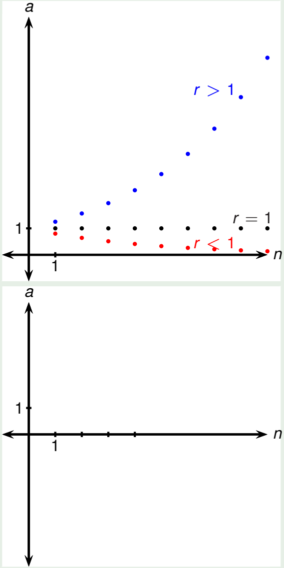

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If $-1 < r < 0$, then $0 < |r| < 1$, and

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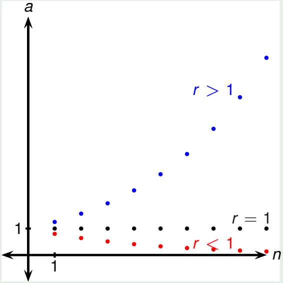

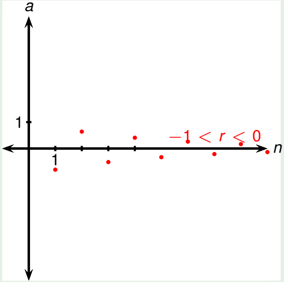

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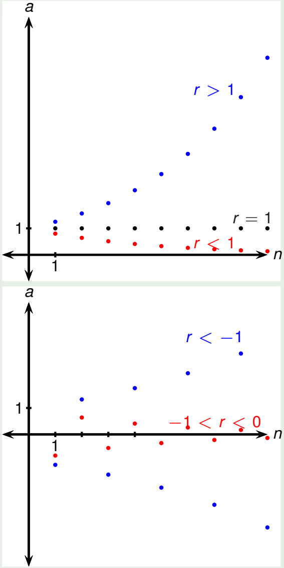
If $r \le -1$, then r^n diverges.



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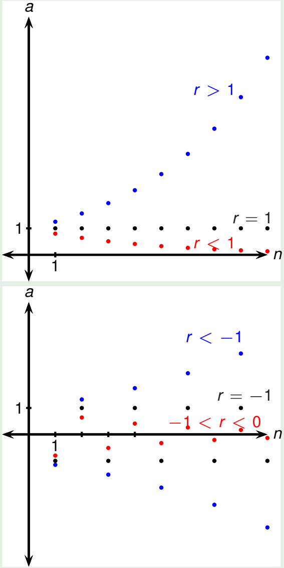
If $r \leq -1$, then r^n diverges. In particular, $(-1)^n$ diverges.



This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Sequences)

The sequence $\{r^n\}$ *is convergent if* $-1 < r \leq 1$ *and divergent otherwise.*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \textit{if} \quad -1 < r < 1 \\ 1 & \textit{if} \quad r = 1 \end{cases}$$

Definition (Increasing and Decreasing)

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is increasing if $a_1 < a_2 < a_3 < \cdots$.
A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is decreasing if $a_1 > a_2 > a_3 > \cdots$.
A sequence is called monotonic if it is either increasing or decreasing.

Example

The sequence $\left\{\frac{1}{2n+1}\right\}$ is decreasing because

$$a_n = \frac{1}{2n+1} \qquad a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$$

and

$$\frac{1}{2n+1} > \frac{1}{2n+3}$$

because the denominator of the latter is bigger.

Definition (Bounded Sequence)

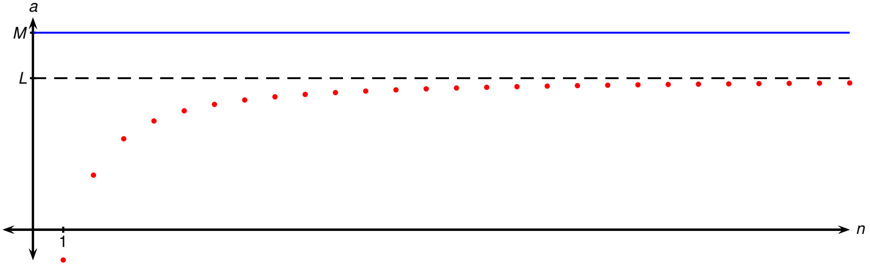
A sequence $\{a_n\}$ is called bounded above if there exists a number M such that

$$a_n < M \qquad \text{for all} \qquad n \geq 1.$$

It is called bounded below if there exists a number M such that

$$a_n > M \qquad \text{for all} \qquad n \geq 1.$$

A bounded sequence is a sequence that is bounded below and above.



Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.