Math 141

Lecture 11

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Outline

Example (A finite series)

$$1 + 2 + 3 + 4 + 5$$

is an example of a finite series. So is

$$-1 - 2 - 3 - \ldots - 10000.$$

Definition (Finite series, Infinite series)

A finite series is a series that ends. It is possible to write down all the terms in a finite series. A series that is not finite is called an infinite series.

Every finite series has a sum. Some infinite series have a sum, and others do not.

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Every finite series has a sum. Some infinite series have a sum, and others do not.

Example (An infinite series)

$$1 + 1 + 1 + 1 + \ldots$$

is an example of an infinite series. It has no sum.

Theorem (Sum of an arithmetic series)

The sum of a finite arithmetic series is the average of the first and last terms, multiplied by the number of terms. That is,

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{a + (a + (n - 1)d)}{2} n.$$

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Therefore the sum is $\frac{5+100}{2} \cdot = 20105 \cdot 10 = 1050$.

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$$\begin{aligned} \text{Therefore} \quad 2s &= (-49)(22) \\ s &= -49 \cdot 22/2 = -539. \end{aligned}$$

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Theorem (The sum of a finite geometric series)

The sum of the finite geometric series $\sum_{n=1}^{M} ar^{n-1}$ *is* $a\frac{1-r^M}{1-r}$.

(12.2) Series

Definition (Series)

If we add the terms in an infinite sequence, we get an infinite series:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

We denote this sum by

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Example (Series notation)

The series

$$2 + 4 + 6 + 8 + \ldots + 124$$

can be written more concisely as

$$\sum_{n=1}^{62} 2 + 2(n-1).$$

$2 + 2(n-1)$ is the nth term, and the sigma sign $\sum$ tell us to add all these terms, starting from $n = 1$ and going up to $n = 62$. In this notation n is called the index.

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Write $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \frac{32}{243} - \frac{64}{729}$ using series notation.

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Example (A series with no sum)

The series $1 + 2 + 3 + 4 + 5 + \cdots$ has no sum.

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- After the nth term, we get

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- After the nth term, we get $\frac{n(n+1)}{2}$.

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- After the nth term, we get $\frac{n(n+1)}{2}$.
- This goes to ∞ as n gets bigger.
- Now consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$$

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- This gets closer and closer to 1. We write $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Definition (Partial Sum, Convergent, Divergent, Sum)

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote the nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s.$$

In this case, we call s the sum of the series.
If the sequence $\{s_n\}$ is divergent, then we say that the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

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- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then r^n is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Series)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the series is divergent.*
a *is called the first term and* r *is called the common ratio.*

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Find the sum of the geometric series

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Write the number $2.3\overline{17} = 2.3171717\ldots$ as a quotient of integers.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

- After the first term, we have a geometric series.
- $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

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Example

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

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$$\text{Therefore} \quad \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \lim_{n\to\infty} s_n$$

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Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)
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\end{aligned}
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Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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\begin{aligned}
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&\vdots \\
s_{2^n} &>
\end{aligned}
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Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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\end{aligned}
$$

Therefore $s_{2^n} \to \infty$ as $n \to \infty$, so $\{s_n\}$ is divergent, so the harmonic series is divergent.