Math 141

Lecture 12

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Outline

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Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

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This is just a restatement of the previous theorem:

Theorem (The Divergence Test)

If $\lim_{n\to\infty} a_n$ *doesn't exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

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Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

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$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

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Therefore, by the Divergence Test, the series diverges.

The Integral Test and Estimates of Sums

- In general, it is not easy to find the sum of a series.
- We could do this for $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ because we found a simple formula for the nth partial sum s_n.
- In the next few sections, we'll learn techniques for showing whether a series is convergent or divergent without explicitly computing its sum.

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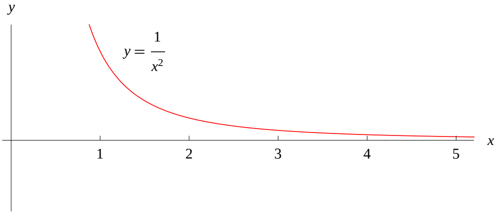

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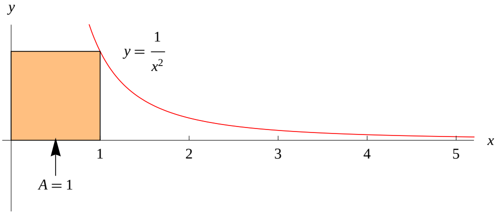

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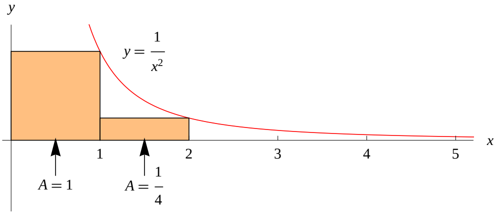

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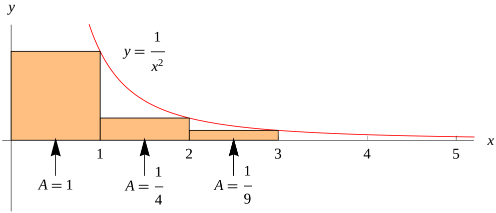

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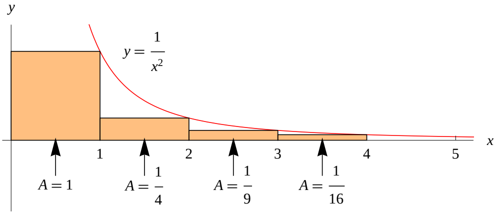

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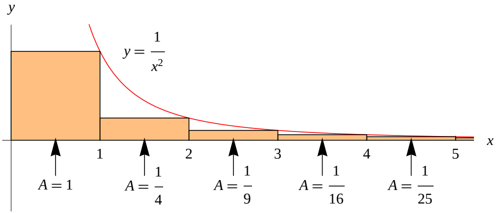

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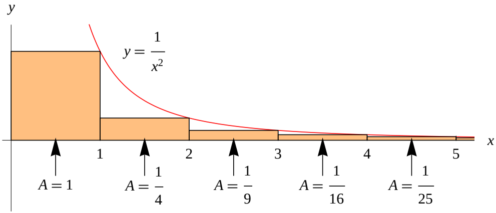

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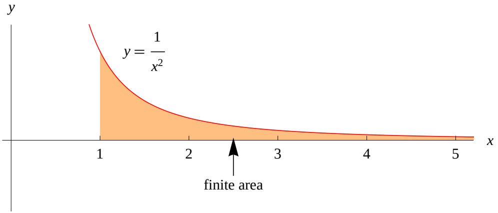

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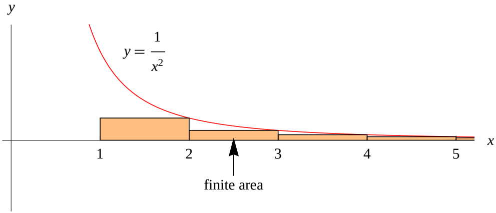

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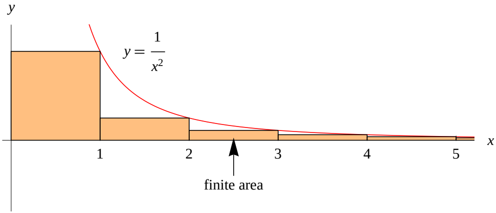

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- Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

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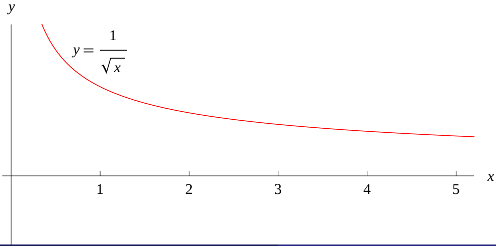

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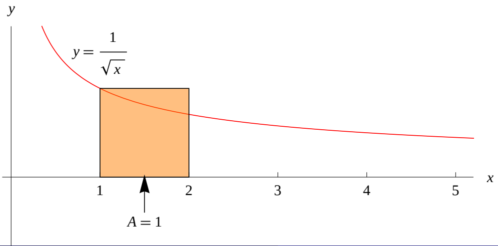

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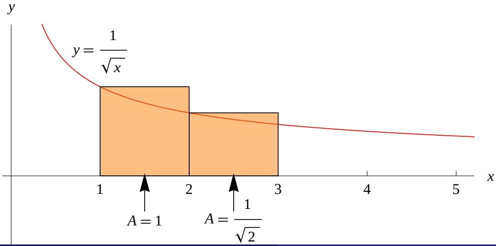

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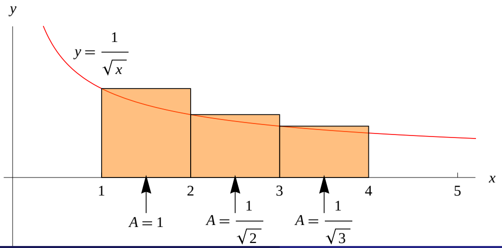

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50	12.7524
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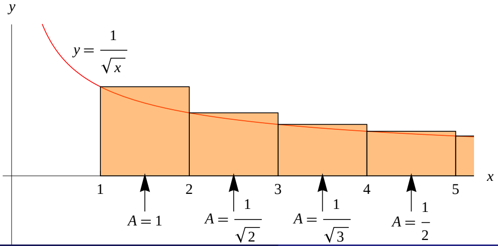

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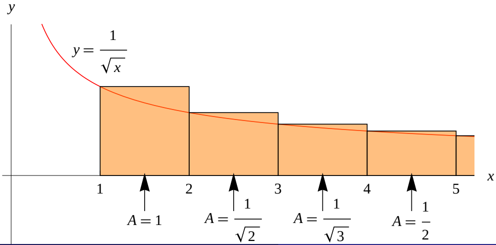

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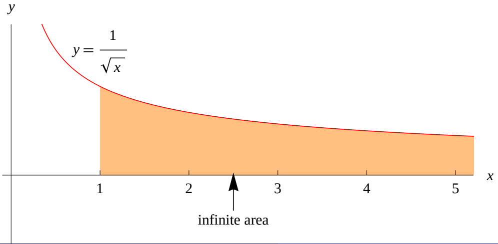

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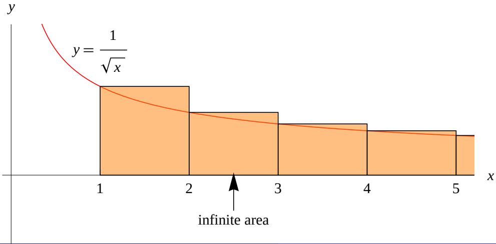

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Theorem (The Integral Test)

Let f be a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent. In other words,

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Note that it is not necessary to start the series or the integral at $n = 1$. For instance, to test the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

we would use

$$\int_4^{\infty} \frac{1}{(x-3)^2}\mathrm{d}x$$

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- $\displaystyle\int_1^{\infty} \frac{1}{x^p} dx$ is divergent if $p \leq 1$.
- Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

This theorem summarizes the results of the previous example.

Theorem (p-series Convergence)

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

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Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

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Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where f is continuous, positive, and decreasing for $x \geq n$, and $\sum a_k$ is convergent with sum s. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\mathrm{d}x$$

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Therefore the error is at most 0.005.

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Therefore the error is at most 0.005.

To get an accuracy of 0.0005 or better, we want $R_n \le 0.0005$. Since $R_n \le \frac{1}{2n^2}$, we want

$$\frac{1}{2n^2} \le 0.0005, \qquad \text{or} \qquad n \ge \sqrt{1000} \approx 31.6$$

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \quad \leq \quad R_n \quad \leq \quad \int_{n}^{\infty} f(x)\mathrm{d}x$$

$$\begin{array}{rcccl} \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \end{array}$$

- Add s_n to both sides of both inequalities.

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- Add s_n to both sides of both inequalities.
- This gives upper and lower bounds for s.
- This is a better approximation than just using s_n.

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- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
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Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
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Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

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- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

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- The Comparison Test tells us nothing here.
- Nevertheless, we think $\sum \frac{1}{2^n-1}$ should converge, because it's so close to $\sum \frac{1}{2^n}$.

Theorem (The Limit Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

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The main thing to check is that c is finite and non-zero.

Example (Example 3, p. 743)

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

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Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
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$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \end{aligned}$$

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&= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}}
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Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0 \end{aligned}$$

- $\sum \frac{1}{2^n}$ is a geometric series.

Example (Example 3, p. 743)

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.
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- $\sum \frac{1}{2^n}$ is a convergent geometric series.
- By the Limit Comparison Test $\sum \frac{1}{2^n-1}$ is convergent too.

Example (Example 4, p. 743)

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ for convergence or divergence.

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$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = 1 > 0$$

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- Therefore $\sum \frac{2}{n^{1/2}}$ is

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- $\sum \frac{2}{n^{1/2}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{1/2}}$ is divergent

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- $\sum \frac{2}{n^{1/2}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{1/2}}$ is divergent, and so is $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$.