Math 141

Lecture 14

Greg Maloney

Todor Milev

University of Massachusetts Boston

Spring 2015

Outline

Outline

Outline

Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.

Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.
- A power series might converge for some values of x and diverge for others.

Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.
- A power series might converge for some values of x and diverge for others.
- The sum of the series is a function

  $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

  whose domain is the set of all x for which the series converges.

Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.
- A power series might converge for some values of x and diverge for others.
- The sum of the series is a function

  $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

  whose domain is the set of all x for which the series converges.
- f resembles a polynomial, except it has infinitely many terms.

Definition (Power Series Centered at a)

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at a or a power series about a or a power series in $(x-a)$.

Definition (Power Series Centered at a)

A series of the form
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$
is called a power series centered at a or a power series about a or a power series in $(x-a)$.

- We use the convention that $(x-a)^0 = 1$, even if $x = a$.

Definition (Power Series Centered at a)

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at a or a power series about a or a power series in $(x-a)$.

- We use the convention that $(x-a)^0 = 1$, even if $x = a$.
- If $x = a$, then all terms are 0 for $n \geq 1$, so the series always converges when $x = a$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}\end{aligned}$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n\to\infty} (n+1) \end{aligned}$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}(n+1)\end{aligned}$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}(n+1)|x|\end{aligned}$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty} (n+1)|x| \\ &= \infty \end{aligned}$$

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}(n+1)|x| \\ &= \infty\end{aligned}$$

- Therefore by the Ratio Test the series diverges for all $x \neq 0$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty} (n+1)|x| \\ &= \infty \end{aligned}$$

- Therefore by the Ratio Test the series diverges for all $x \neq 0$.
- Therefore the series only converges for $x = 0$.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty}$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3|$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty} |x-3|$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \end{aligned}$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}}$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3| \end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if
  and diverges if

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}|x-3|\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}|x-3|\frac{1}{1+\frac{1}{n}} = |x-3|\end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3|<1$ and diverges if

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| \\ &= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3| \end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| \\
&= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|
\end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| \\ &= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3| \end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is
- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.
- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}|x-3|\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}|x-3|\frac{1}{1+\frac{1}{n}} = |x-3|\end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3|<1$ and diverges if $|x-3|>1$.

$$|x-3|<1 \quad\Leftrightarrow\quad -1<x-3<1 \quad\Leftrightarrow\quad 2<x<4$$

- If we put $x=4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.
- If we put $x=2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}|x-3|\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}|x-3|\frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.
- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is convergent.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.
- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is convergent.
- The series converges if $2 \leq x < 4$ and diverges otherwise.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right|$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \lim_{n\to\infty} \underline{\qquad\qquad}\end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{}\end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\
&= \lim_{n\to\infty}\frac{x^2}{}
\end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{4}\end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{4} \end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} \end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 \end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1 \end{aligned}$$

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1 \end{aligned}$$

- Therefore by the Ratio Test the series converges for all x.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\
&= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1
\end{aligned}$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore the domain of the function is $(-\infty, \infty)$, or $\mathbb{R}$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, *there are three possibilities:*

1. *The series converges only when* $x = a$.
2. *The series converges for all* x.
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, *there are three possibilities:*

1. *The series converges only when* $x = a$.
2. *The series converges for all* x.
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Definition (Radius of Convergence)

The number R in case three of the theorem is called the radius of convergence of the power series.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$*, there are three possibilities:*

1. *The series converges only when* $x = a$*.*
2. *The series converges for all* x*.*
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Definition (Radius of Convergence)

The number R in case three of the theorem is called the radius of convergence of the power series.

1. In the first case, we say $R = 0$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

Definition (Radius of Convergence)

The number R in case three of the theorem is called the radius of convergence of the power series.

1. In the first case, we say $R = 0$.
2. In the second case, we say $R = \infty$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, *there are three possibilities:*

1. *The series converges only when* $x = a$.
2. *The series converges for all* x.
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$*, there are three possibilities:*

1. *The series converges only when* $x = a$*.*
2. *The series converges for all* x*.*
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

1. In the first case, the interval contains the single point a.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

1. In the first case, the interval contains the single point a.
2. In the second case, the interval is $(-\infty, \infty)$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

1. In the first case, the interval contains the single point a.
2. In the second case, the interval is $(-\infty, \infty)$.
3. In the third case, the inequality $|x-a| < R$ can be rewritten $a - R < x < a + R$.

What happens at the endpoints of the interval $a - R < x < a + R$?

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
  1. $[a - R, a + R)$
  2. $(a - R, a + R]$

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
  1. $[a - R, a + R)$
  2. $(a - R, a + R]$
  3. $[a - R, a + R]$

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
  1. $[a - R, a + R)$
  2. $(a - R, a + R]$
  3. $[a - R, a + R]$
  4. $(a - R, a + R)$

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
    1. $[a - R, a + R)$
    2. $(a - R, a + R]$
    3. $[a - R, a + R]$
    4. $(a - R, a + R)$
- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
    1. $[a - R, a + R)$
    2. $(a - R, a + R]$
    3. $[a - R, a + R]$
    4. $(a - R, a + R)$
- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.
- The Ratio and Root Tests will always fail when x is an endpoint $a - R$ or $a + R$, so the endpoints must be checked with another test.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3
\end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\ &= \lim_{n\to\infty} 3 \end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty} 3|x|$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \lim_{n\to\infty} 3|x|$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\ &= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}
\end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\
&= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}
\end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}$$

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if and diverges if

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\
&= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\ &= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent.
- The interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a =$ and $r =$

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r =$

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r =$

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if and divergent otherwise.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{}{1-}$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{1}{1-}$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{1}{1-x}$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{1}{1-x}$.
- The thing that is new in this section is the we now regard the series $\sum_{n=0}^{\infty} x^n$ as expressing the function $f(x) = \frac{1}{1-x}$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{1}{1-x}$.
- The thing that is new in this section is the we now regard the series $\sum_{n=0}^{\infty} x^n$ as expressing the function $f(x) = \frac{1}{1-x}$.
- This only works if $-1 < x < 1$.

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty} (-x^2)^n &&= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \\
& &&= 1 - x^2 + x^4 - x^6 + \cdots
\end{aligned}$$

- Another way to write the series is $\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

- Another way to write the series is $\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.
- This converges if $|-x^2| < 1$, that is, if $x^2 < 1$, or $|x| < 1$.

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots \\ \frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty} (-x^2)^n &&= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \\ & &&= 1 - x^2 + x^4 - x^6 + \cdots \end{aligned}$$

- Another way to write the series is $\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.
- This converges if $|-x^2| < 1$, that is, if $x^2 < 1$, or $|x| < 1$.
- Therefore the interval of convergence is $(-1, 1)$.

Example

Find a power series representation for $\frac{1}{x+2}$.

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n} = \sum_{n=0}^{\infty} \underline{\qquad}\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^{n}}{}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^{n}}{}\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{}x^n\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{}x^n$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\ &= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots\end{aligned}$$

Interval of convergence:

$$< \quad 1$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots\end{aligned}$$

Interval of convergence:

$$\left|-\frac{x}{2}\right| < 1$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots\end{aligned}$$

Interval of convergence:

$$\begin{aligned}\left|-\frac{x}{2}\right| &< 1\\ |x| &< 2\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots\end{aligned}$$

Interval of convergence:

$$\begin{aligned}\left|-\frac{x}{2}\right| &< 1\\ |x| &< 2\end{aligned}$$

Therefore the interval of convergence is $(-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2}$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2}$$

$$= x^3$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned} \frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\ &= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \end{aligned}$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$
\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}
\end{aligned}
$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}
\end{aligned}$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
\end{aligned}$$

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
\end{aligned}$$

- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
\end{aligned}$$

- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.
- The interval of convergence is again $(-2, 2)$.

Differentiation and Integration of Power Series

Theorem (Differentiation and Integration of Power Series)

If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

1. $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$
2. $$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$
   $= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$

- This is called term-by-term differentiation and integration.
- Another way of saying it is

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

$$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right] dx$$

- We can treat power series like polynomials with infinitely many terms.

Example

Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Example

Find the derivative of the Bessel function

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \end{aligned}$$

Example

Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$

Example

Find the derivative of the Bessel function

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} \end{aligned}$$

- $J_0(x)$ is defined

Example

Find the derivative of the Bessel function

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} \end{aligned}$$

- $J_0(x)$ is defined everywhere.

Example

Find the derivative of the Bessel function

$$
\begin{aligned}
J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\
J_0'(x) &= \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}
\end{aligned}
$$

- $J_0(x)$ is defined everywhere.
- Therefore its derivative $J_0'(x)$ is also defined everywhere.

Example

Find a power series for $\ln(1 - x)$ and state its radius of convergence.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) =$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\ln(1-x) = -\int \frac{1}{1-x}\mathrm{d}x$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\ln(1-x) = -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx \\ &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)\end{aligned}$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}
\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx \\
&= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right) + C
\end{aligned}$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}
\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1 + x + x^2 + x^3 + \cdots)dx \\
&= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) + C = C - \sum_{n=1}^{\infty} \frac{1}{n}x^n
\end{aligned}$$

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned} \ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx \\ &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)+C = C-\sum_{n=1}^{\infty}\frac{1}{n}x^n \end{aligned}$$

- The radius is the same as for the original series: $R=1$.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx \\ &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)+C = C-\sum_{n=1}^{\infty}\frac{1}{n}x^n\end{aligned}$$

- The radius is the same as for the original series: $R=1$.
- To find C, plug in $x=0$:

Example

Find a power series for $\ln(1 - x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}
\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1 + x + x^2 + x^3 + \cdots)dx \\
&= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) + C = C - \sum_{n=1}^{\infty}\frac{1}{n}x^n
\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

Therefore

$$\begin{aligned}\ln(1-x) &= -\int \frac{1}{1-x}dx = -\int (1+x+x^2+x^3+\cdots)dx \\ &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)+C = C-\sum_{n=1}^{\infty}\frac{1}{n}x^n\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

Therefore

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{1}{n}x^n$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x =$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1+x^2}$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\arctan x = \int \frac{1}{1+x^2}dx$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\arctan x = \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)\end{aligned}$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned} \arctan x &= \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \cdots) dx \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) + C \end{aligned}$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\
&= C + \sum_{n=0}^{\infty} \qquad \overline{}
\end{aligned}$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\mathrm{d}x \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \underline{\qquad\qquad}
\end{aligned}$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \underline{\qquad\qquad}
\end{aligned}$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\mathrm{d}x \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x^2}\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\mathrm{d}x \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\ &= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$:

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned} \arctan x &= \int \frac{1}{1+x^2}\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\mathrm{d}x \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

Therefore

$$\begin{aligned}\arctan x &= \int \frac{1}{1+x^2}\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\mathrm{d}x \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C \\ &= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) =$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) =$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5c_5(x-a)^3 + \cdots$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) =$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) =$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) = 2 \cdot 3c_3 = 3!c_3$.

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) = 2 \cdot 3c_3 = 3!c_3$.
- $f^{(n)}(a) =$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) = 2 \cdot 3c_3 = 3!c_3$.
- $f^{(n)}(a) = n!c_n$.

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) = 2 \cdot 3c_3 = 3!c_3$.
- $f^{(n)}(a) = n!c_n$.
- Therefore $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Here is what we get if we plug these coefficients into the power series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Here is what we get if we plug these coefficients into the power series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Definition (Taylor Series)

This series is called the Taylor series of f.

The case when $a = 0$ is special enough to have its own name:

Definition (Maclaurin Series)

The Maclaurin series of f is the Taylor series of f centered at $a = 0$. In other words, it is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) =$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) =$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{\quad}{\quad}$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{}$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{\quad}$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1}$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} =$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

- Therefore by the Ratio Test the series converges for all x.

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n n!} = 1 - \frac{1}{2 \cdot 1!} + \frac{1}{4 \cdot 2!} - \frac{1}{8 \cdot 3!} + \cdots$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\qquad\right)^n$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \frac{1}{8\cdot 3!} + \cdots$$

$$\begin{aligned} e^{x} &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n \\ \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n \\ &= e^{-1/2} \end{aligned}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n n!} = 1 - \frac{1}{2 \cdot 1!} + \frac{1}{4 \cdot 2!} - \frac{1}{8 \cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n n!} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{2}\right)^n \\ &= e^{-1/2} \\ &= \frac{1}{\sqrt{e}} \end{aligned}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) =$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) =$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n}\right|$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^3 (x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3 (x-3)^n} \right| = \lim_{n\to\infty} \underline{\qquad}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty} \frac{|x-3|}{n+1}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} =$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n} \right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$
- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.
- Just like the Maclaurin series, this series also represents e^x.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$e^x = e^{x-3+3}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$e^x = e^{x-3+3} = e^3 e^{x-3}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$e^x = e^{x-3+3} = e^3 e^{x-3}$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$e^x = e^{x-3+3} = e^3 e^{x-3}$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$\begin{aligned} e^x &= e^{x-3+3} = e^3 e^{x-3} \\ &= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \end{aligned}$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$\begin{aligned} e^x &= e^{x-3+3} = e^3 e^{x-3} \\ &= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n \end{aligned}$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$\begin{aligned} e^x &= e^{x-3+3} = e^3 e^{x-3} \\ &= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n \end{aligned}$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

The radius of convergence was already computed to be $R = \infty$.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{aligned}
f(x) &= \sin x & \qquad f(0) &= \\
f'(x) &= & f'(0) &= \\
f''(x) &= & f''(0) &= \\
f'''(x) &= & f'''(0) &= \\
f^{(4)}(x) &= & f^{(4)}(0) &=
\end{aligned}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= & f'(0) &= \\ f''(x) &= & f''(0) &= \\ f'''(x) &= & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= & f'(0) &= \\
f''(x) &= & f''(0) &= \\
f'''(x) &= & f'''(0) &= \\
f^{(4)}(x) &= & f^{(4)}(0) &=
\end{aligned}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= \\
f''(x) &= & f''(0) &= \\
f'''(x) &= & f'''(0) &= \\
f^{(4)}(x) &= & f^{(4)}(0) &=
\end{aligned}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= \\ f''(x) &= & f''(0) &= \\ f'''(x) &= & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{array}{rclcrcl}
f(x) & = & \sin x & & f(0) & = & 0 \\
f'(x) & = & \cos x & & f'(0) & = & 1 \\
f''(x) & = & & & f''(0) & = & \\
f'''(x) & = & & & f'''(0) & = & \\
f^{(4)}(x) & = & & & f^{(4)}(0) & = &
\end{array}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= & f''(0) &=\\
f'''(x) &= & f'''(0) &=\\
f^{(4)}(x) &= & f^{(4)}(0) &=
\end{aligned}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= \\ f'''(x) &= & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= \\ f'''(x) &= & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{array}{rclcrcl}
f(x) &=& \sin x & & f(0) &=& 0 \\
f'(x) &=& \cos x & & f'(0) &=& 1 \\
f''(x) &=& -\sin x & & f''(0) &=& 0 \\
f'''(x) &=& & & f'''(0) &=& \\
f^{(4)}(x) &=& & & f^{(4)}(0) &=&
\end{array}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= \\ f^{(4)}(x) &= & f^{(4)}(0) &= \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f'''(x) &= -\cos x & f'''(0) &= -1\\
f^{(4)}(x) &= & f^{(4)}(0) &=
\end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& & f^{(4)}(0) &=& \end{array}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{array}{rclrcl}
f(x) & = & \sin x & f(0) & = & 0 \\
f'(x) & = & \cos x & f'(0) & = & 1 \\
f''(x) & = & -\sin x & f''(0) & = & 0 \\
f'''(x) & = & -\cos x & f'''(0) & = & -1 \\
f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = &
\end{array}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & \end{array}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{array}{rclrcl}
f(x) & = & \sin x & f(0) & = & 0 \\
f'(x) & = & \cos x & f'(0) & = & 1 \\
f''(x) & = & -\sin x & f''(0) & = & 0 \\
f'''(x) & = & -\cos x & f'''(0) & = & -1 \\
f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0
\end{array}
$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclcrcl} f(x) & = & \sin x & & f(0) & = & 0 \\ f'(x) & = & \cos x & & f'(0) & = & 1 \\ f''(x) & = & -\sin x & & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \qquad \underline{\qquad\qquad}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & f^{(4)}(0) &=& 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \underline{\qquad\qquad}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \underline{\qquad\qquad}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \underline{\qquad\qquad} \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{} \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclcrcl} f(x) & = & \sin x & & f(0) & = & 0 \\ f'(x) & = & \cos x & & f'(0) & = & 1 \\ f''(x) & = & -\sin x & & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{} \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) & = & \sin x & f(0) & = & 0 \\ f'(x) & = & \cos x & f'(0) & = & 1 \\ f''(x) & = & -\sin x & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & f'''(0) & = & -1 \\ f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Therefore $R =$.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Therefore $R = \infty$.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & f^{(4)}(0) &=& 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

$$\sin x \quad = \quad \sum_{n=0}^{\infty}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8 \cdot 3!} + \frac{\pi^5}{32 \cdot 5!} - \frac{\pi^7}{128 \cdot 7!} + \cdots$$

$$\sin x \quad = \quad \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} \left(\quad \right)^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8 \cdot 3!} + \frac{\pi^5}{32 \cdot 5!} - \frac{\pi^7}{128 \cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}$$

$$= \sin\frac{\pi}{2}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8 \cdot 3!} + \frac{\pi^5}{32 \cdot 5!} - \frac{\pi^7}{128 \cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1} \\ &= \sin\frac{\pi}{2} \\ &= 1\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{d}{dx}\left(\qquad \right)$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{d}{dx}\left(\sin x\right)$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\qquad\qquad\qquad \right)$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{}
\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{d}{dx}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$x \cos x = x$$

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{}{(2n)!} \end{aligned}$$

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \end{aligned}$$

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots \end{aligned}$$

Here is a table of some important Maclaurin series we have learned:

Function	Series	R
$\dfrac{1}{1-x}$	$= \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	1
$\arctan x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	1
e^x	$= \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	∞
$\sin x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞

Example (Example 11, p. 780)

Use a power series to find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Example (Example 11, p. 780)

Use a power series to find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example (Example 11, p. 780)

Use a power series to find $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

Example (Example 11, p. 780)

Use a power series to find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \end{aligned}$$

Example (Example 11, p. 780)

Use a power series to find $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right)
\end{aligned}$$

Example (Example 11, p. 780)

Use a power series to find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) =
\end{aligned}$$

Example (Example 11, p. 780)

Use a power series to find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \frac{1}{2}
\end{aligned}
$$

Example

Use a power series to find $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

Example

Use a power series to find $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example

Use a power series to find $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ -\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \end{aligned}$$

Example

Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ -\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\ x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \end{aligned}$$

Example

Use a power series to find $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ -\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\ x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\ \frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \end{aligned}$$

Example

Use a power series to find $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
-\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right)
\end{aligned}$$

Example

Use a power series to find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
-\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) =
\end{aligned}$$

Example

Use a power series to find $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$.

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
-\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) = \frac{1}{6}
\end{aligned}
$$