Freecalc
Homework on Lecture 9
Quiz date will be announced in class

1. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int_{2}^{\infty} \frac{1}{(x-1)^{\frac{3}{2}}}\mathrm{d}x.$ answer: convergent

(b) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$ answer: convergent

(c) $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$ answer: divergent

(d) $\displaystyle\int_{-1}^{\infty} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$ answer: divergent

(e) $\displaystyle\int_{-\infty}^{0} \frac{1}{2-3x}\mathrm{d}x.$ answer: divergent

(f) $\displaystyle\int_{-\infty}^{0} \frac{1}{(2-3x)^2}\mathrm{d}x.$ answer: convergent

(g) $\displaystyle\int_{-\infty}^{0} \frac{1}{(2-3x)^{1.00000001}}\mathrm{d}x.$ answer: convergent

(h) $\displaystyle\int_{-2}^{\frac{1}{2}} \frac{1}{2x-1}\mathrm{d}x.$ answer: divergent

(i) $\displaystyle\int_{-5}^{\infty} e^{-3x}\mathrm{d}x.$ answer: convergent

(j) $\displaystyle\int_{-\infty}^{5} 2^x\mathrm{d}x.$ answer: convergent

(k) $\displaystyle\int_{-\infty}^{\infty} x^3\mathrm{d}x.$ answer: divergent

(l) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\mathrm{d}x.$ answer: convergent

(m) $\displaystyle\int_{0}^{\infty} \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x.$ answer: convergent

(n) $\displaystyle\int_{0}^{\infty} \sin^2 x\mathrm{d}x.$ answer: divergent

(o) $\displaystyle\int_{0}^{5} \frac{1}{x^2+x-2}\mathrm{d}x.$ answer: divergent

(p) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+x+1}\mathrm{d}x.$ answer: convergent

(q) $\displaystyle\int_{2}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$ answer: convergent

(r) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$ answer: divergent

(s) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+2}\mathrm{d}x.$ answer: convergent

Solution. 1.s The integrand is a rational function and therefore we can solve this problem by finding the indefinite integral and then computing the limit. We would need to start by factoring x^4+2 into irreducible quadratic factors - that is already quite laborious:

$$x^4+2 = \left(x^2+\sqrt[4]{8}x+\sqrt{2}\right)\left(x^2-\sqrt[4]{8}x+\sqrt{2}\right) \quad .$$

The problem asks us only to establish the convergence of the integral; it does not ask us to compute its actual numerical value. Therefore we can give a much simpler solution. The function is even and therefore it suffices to establish whether $\displaystyle\int_{0}^{\infty} \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

We have that

$$\int_0^\infty \frac{x^2}{x^4+2}\mathrm{d}x = \int_0^1 \frac{x^2}{x^4+2}\mathrm{d}x + \int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \quad .$$

The function $\frac{x^2}{x^4+2}$ is continuous so $\int_0^1 \frac{x^2}{x^4+2}\mathrm{d}x$ integrates to a number, which does not affect the convergence of the above expression. Therefore the convergence of our integral is governed by the convergence of $\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$. To establish that that integral is convergent, we use the comparison theorem as follows.

$$\begin{aligned} \int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x &\leq \int_1^\infty \frac{x^2}{x^4}\mathrm{d}x && \Big| \text{ we have that } x^4+2 > x^4 \text{ and therefore } \frac{x^2}{x^4+2} \leq \frac{x^2}{x^4} \\ &= \int_1^\infty x^{-2}\mathrm{d}x \\ &= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t \\ &= \lim_{t\to\infty} 1 - \frac{1}{t} \\ &= 1 \quad . \end{aligned}$$

In this way we showed $\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \leq 1$. Therefore, as $\frac{x^2}{x^4+2} \geq 0$ is positive, we can apply the comparison theorem to get that $\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

Solution. 1.m It is possible to show that this integral is convergent by using the comparison theorem. However, we shall use direct integration instead. First, we solve the indefinite integral:

$$\begin{aligned} \int \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x &= \int \sqrt{x}e^{-\sqrt{x}}\frac{2\sqrt{x}\mathrm{d}x}{2\sqrt{x}} && \Big| \text{ use } \mathrm{d}\sqrt{x} = \tfrac{\mathrm{d}x}{2\sqrt{x}} \\ &= \int \sqrt{x}e^{-\sqrt{x}}\left(2\sqrt{x}\mathrm{d}\sqrt{x}\right) && \Big| \text{ Set } \sqrt{x} = u \\ &= 2\int u^2e^{-u}\mathrm{d}u \\ &= 2\left(-\int u^2\mathrm{d}\left(e^{-u}\right)\right) && \Big| \text{ integrate by parts} \\ &= 2\left(-u^2e^{-u} + \int e^{-u}\mathrm{d}\left(u^2\right)\right) \\ &= 2\left(-u^2e^{-u} + \int 2ue^{-u}\mathrm{d}u\right) \\ &= 2\left(-u^2e^{-u} - \int 2u\mathrm{d}e^{-u}\right) && \Big| \text{ integrate by parts again} \\ &= 2\left(-u^2e^{-u} - 2ue^{-u} + \int 2e^{-u}\mathrm{d}u\right) \\ &= 2\left(-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right) + C \\ &= 2\left(-xe^{-\sqrt{x}} - 2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right) + C \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x &= \lim_{t\to\infty} 2\left[-xe^{-\sqrt{x}} - 2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right]_0^\infty \\ &= 4 + \lim_{t\to\infty} 4\left(-te^{-\sqrt{t}} - \sqrt{t}e^{-\sqrt{t}} - e^{-\sqrt{t}}\right) && \Big| \text{ Set } u = \sqrt{t} \\ &= 4 - 4\lim_{u\to\infty}\left(u^2e^{-u} + ue^{-u} + e^{-u}\right) \\ &= 4 - 4\lim_{u\to\infty}\frac{u^2+u+1}{e^u} && \Big| \text{ use L'Hospital's rule for limit, see below} \\ &= 4 \quad , \end{aligned}$$

and the integral converges to 4. In the above computation we used the following limit computation

$$\begin{aligned} \lim_{u\to\infty} \frac{u^2+u+1}{e^u} &= \lim_{u\to\infty} \frac{2u+1}{e^u} && \Big| \text{ Apply L'Hospital's rule} \\ &= \lim_{u\to\infty} \frac{2}{e^u} \\ &= 0 \quad . \end{aligned}$$

2. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int_{100}^{\infty} \frac{1}{x\ln x} \mathrm{d}x.$ answer: divergent

(b) $\displaystyle\int_{100}^{\infty} \frac{1}{x(\ln x)^2} \mathrm{d}x.$ answer: convergent

(c) $\displaystyle\int_{0}^{1} \ln x \mathrm{d}x.$ answer: convergent

(d) $\displaystyle\int_{0}^{1} \frac{\ln x}{\sqrt{x}} \mathrm{d}x.$ answer: convergent

(e) $\displaystyle\int_{0}^{2} x^3 \ln x \mathrm{d}x.$ answer: convergent

(f) $\displaystyle\int_{0}^{1} \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$ answer: divergent

(g) $\displaystyle\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$ answer: convergent

(h) $\displaystyle\int_{0}^{\infty} \sin x^2 \mathrm{d}x$ (This problem is more difficult and may require knowledge of sequences to solve). answer: convergent