Freecalc
Homework on Lecture 8
Quiz date will be announced in class

1. Compute the limits. The answer key has not been fully proofread, use with caution.

   (a) $\lim\limits_{x\to 0} \frac{\sin x}{x}$. answer: 1

   (b) $\lim\limits_{x\to 0} \frac{x}{\ln(1+x)}$. answer: 1

   (c) $\lim\limits_{x\to 0} \frac{x^2}{x-\ln(1+x)}$. answer: 2

   (d) $\lim\limits_{x\to 0} \frac{x^2}{\sin x \ln(1+x)}$. answer: 1

   (e) $\lim\limits_{x\to 0} \frac{\sin^2 x}{(\ln(1+x))^2}$. answer: 1

   (f) $\lim\limits_{x\to 0} \frac{\cos x - 1}{\sin x \ln(1+x)}$. answer: $-\frac{1}{2}$

   (g) $\lim\limits_{x\to 0} \frac{\arctan x - x}{x^3}$. answer: $-\frac{1}{3}$

   (h) $\lim\limits_{x\to 0} \frac{\arcsin x - x}{x^3}$. answer: $\frac{1}{6}$

   (i) $\lim\limits_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$. answer: $\frac{1}{2}$

   (j) $\lim\limits_{x\to 0} \frac{\cos(nx)-\cos(mx)}{x^2}$. answer: $\frac{m^2-n^2}{2}$

   (k) (Optional) $\lim\limits_{x\to 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$. answer: $\frac{3}{40}$

   The very last problem can be done easily using Maclaurin series, but we challenge the student to try it using L'Hospital's rule.

2. Find the limit.

   (a) $\lim\limits_{x\to 0} \frac{\sin x - x}{\arcsin x - x}$. answer: -1

   (b) $\lim\limits_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x)+1}$. answer: $-\frac{2}{\pi}$

   (c) $\lim\limits_{x\to 0} \frac{\sin x - x}{\arctan x - x}$. answer: $\frac{1}{2}$

   (d) $\lim\limits_{x\to \infty} x \sin\left(\frac{2}{x}\right)$. answer: 2

Solution. 2.b The limit is of the form "$\frac{0}{0}$" so we are allowed to use L'Hospital's rule.

$$
\begin{aligned}
\lim_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x)+1} &= \lim_{x\to 1} \frac{(\sin(\pi x)\ln x)'}{(\cos(\pi x)+1)'} \\
&= \lim_{x\to 1} \frac{\left(\pi\cos(\pi x)\ln x + \sin(\pi x)\frac{1}{x}\right)}{(-\pi\sin(\pi x))} \qquad \text{type "}\tfrac{0}{0}\text{", L'Hospital's rule} \\
&= \lim_{x\to 1} \frac{\left(\pi\cos(\pi x)\ln x + \sin(\pi x)\frac{1}{x}\right)'}{(-\pi\sin(\pi x))'} \\
&= \lim_{x\to 1} \frac{-\pi^2\sin(\pi x)\ln(x) + 2\pi\cos(\pi x)x^{-1} - \sin(\pi x)x^{-2}}{(-\pi^2\cos(\pi x))} \\
&= \frac{-\pi^2\sin(\pi)\ln(1) + 2\pi\cos(\pi) - \sin(\pi)}{(-\pi^2\cos(\pi))} \\
&= -\frac{2}{\pi} .
\end{aligned}
$$

Solution. 2.c **Solution I.**

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sin x - x}{\arctan x - x} &= \lim_{x\to 0}\frac{\cos x - 1}{\frac{1}{1+x^2}-1} && \text{L'Hospital rule}\\
&= \lim_{x\to 0}\frac{-\sin x}{\frac{-2x}{(1+x^2)^2}} && \text{L'Hospital rule again}\\
&= \lim_{x\to 0}\frac{(1+x^2)^2}{2}\frac{\sin x}{x}\\
&= \lim_{x\to 0}\frac{(1+x^2)^2}{2}\lim_{x\to 0}\frac{\sin x}{x}\\
&= \frac{1}{2}\quad .
\end{aligned}
$$

Solution II.

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sin x - x}{\arctan x - x} &= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)-x}{\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)-x} && \text{use the Maclaurin series of } \sin, \arctan\\
&= \lim_{x\to 0}\frac{-\frac{x^3}{6}+x^5\left(\frac{1}{5!}-\dots\right)}{-\frac{x^3}{3}+x^5\left(\frac{1}{5}-\dots\right)} && \text{The expressions in parenthesis are continous functions in x}\\
&= \lim_{x\to 0}\frac{-\frac{1}{6}+x^2\left(\frac{1}{5!}-\dots\right)}{-\frac{1}{3}+x^2\left(\frac{1}{5}-\dots\right)}\\
&= \frac{-\frac{1}{6}+0}{\frac{1}{3}+0}\\
&= \frac{1}{2}\quad .
\end{aligned}
$$

Solution. 2.d.

$$
\begin{aligned}
\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right) &= \lim_{x\to\infty}\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} && \text{indeterminate form Use L'Hospital's rule}\\
&= \lim_{x\to\infty}\frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty} 2\cos\left(\frac{x}{2}\right)\\
&= 2\quad .
\end{aligned}
$$

1 The rest of the problems will not appear on the quiz

3.

4. Compute the limit.

(a) $\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x$.

answer: e^{-2}

(b) $\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^{2x}$

answer: e^{-4}.

(c) $\lim_{x\to\infty}\left(\frac{x}{x+3}\right)^{2x}$

answer: e^{-6}

Solution. 4.a. Solution I

$$
\begin{aligned}
\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x &= \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x && \text{use } \lim_{x\to\infty}\left(1+\tfrac{k}{x}\right)^x = e^k\\
&= e^{-2}\quad .
\end{aligned}
$$

4.a. Solution II

$$\begin{array}{rcll}
\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^x & = & \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^x\right)} & \\
\lim\limits_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^x\right) & = & \lim\limits_{x\to\infty} x\left(\ln(x-2)-\ln(x)\right) & \\
& = & \lim\limits_{x\to\infty}\frac{\ln(x-2)-\ln(x)}{\frac{1}{x}} & \Big| \text{ L'Hospital rule} \\
& = & \lim\limits_{x\to\infty}\frac{\frac{1}{x-2}-\frac{1}{x}}{-\frac{1}{x^2}} & \\
& = & \lim\limits_{x\to\infty}\frac{-2x^2}{x^2-2x} = -2 & \Big| \text{ Therefore} \\
\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^x & = & \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^x\right)} & \\
& = & e^{\lim\limits_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^x\right)} & \\
& = & e^{-2} \quad . &
\end{array}$$

5. Find the limit.

(a) $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^x$. answer: e^{-2}

(b) $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}}$. answer: e^{-1}

(c) $\lim\limits_{x\to\infty}\left(\frac{x}{x-5}\right)^x$. answer: e^5

(d) $\lim\limits_{x\to\infty}\left(\frac{x}{x-2}\right)^{3x+2}$. answer: e^6

6. (a) A sum is held under a yearly compound interest of 1%. Make an approximation by hand (no calculators allowed) by what factor will have the money increased after 200 years. Can you do the computation in your head?

(b) Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years?

Solution. 6.a Each year, the sum increases by a factor of $\left(1+\frac{1}{100}\right)$. Therefore in 200 years the sum will have increased by

$$\begin{array}{rcll}
\left(1+\frac{1}{100}\right)^{200} & = & \left(\left(1+\frac{1}{100}\right)^{100}\right)^2 & \Big| \text{ equals } \left(1+\frac{1}{n}\right)^n \text{ for } n=100 \\
& \approx & e^2. &
\end{array}$$

As a rough estimate for e we can take $e\approx 2.7$, and so $e^2\approx 2.7^2 = 7.29$. Our sum will have increased approximately 7.3 times. A calculator computation shows that

$$\left(1+\frac{1}{100}\right)^{200}\approx 7.316018,$$

so our "in the head" estimate is fairly accurate. Notice that the calculator computation is on its own an approximation - it was carried using double floating point precision arithmetics, which does introduce some minimal errors. Such round off errors, of course, are also present in modern banking transactions, so we do not need to adjust for those.

Solution. 6.b Simple interest of 11% per 150 years a profit of

$$0.11 * 150 = 15 + 1.5 = 16.5,$$

or altogether 17.5-fold increase of our initial sum. A 2% compound interest for 150 years yields a

$$\begin{array}{rcl}
\left(1+\frac{2}{100}\right)^{150} & = & \left(\left(1+\frac{1}{50}\right)^{50}\right)^3 \\
& \approx & e^3
\end{array}$$

-fold increase of our sum. To establish which of the two options yields more money, we need to compare e^3 to 17.5 (without using a calculator). In the solution of 6.a we established that $e^2\approx 7.3$, so $e^3\approx e\cdot 7.3\approx 2.7\cdot 7.3 = 2\cdot 7 + 2\cdot 0.3 + 0.7\cdot 7 + 0.7\cdot 0.3 = 14 + 0.6 + 4.9 + 0.21 = 19.71\approx 19.7$. We can say that the compound interest results in approximately 19.7-fold increase of the initial sum, so the compound interest is more profitable. A calculator computation shows that

$$\left(1+\frac{2}{100}\right)^{150}\approx 19.499603 \quad .$$

Our error of approximately 0.2 was not optimal, yet fairly accurate for an "in the head" computation.

7. $1,000,000$ servers are handling Internet users. Suppose we distribute the computation load as follows. The computation load distributing program directs every new user to a server chosen at random (one server is allowed to process more than one user at a time). Suppose one server has defective hardware and crashes. We are testing the system by simulating X Internet users.

   - What is the chance we catch the defective server using 1 simulated user?
   - Without using a calculator, estimate the chance we fail to catch the defective server using $1,000,000$ simulated users.
   - Using a calculator, estimate the chance we fail to catch the defective server using $100,000$ simulated users. Write an expression using e which approximates this chance. Evaluate the latter with a calculator. Are the two numbers close?

   Remark. While such a simple system architecture would not be practical, it is not to be immediately dismissed as terrible. For example, if we need to handle 2 million users per second, our load distributing mechanism might not be fast enough to keep track of each server's load. On the other hand, an inexpensive modern pc will easily generate 2 million random numbers per second.