Math 141

Lecture 1

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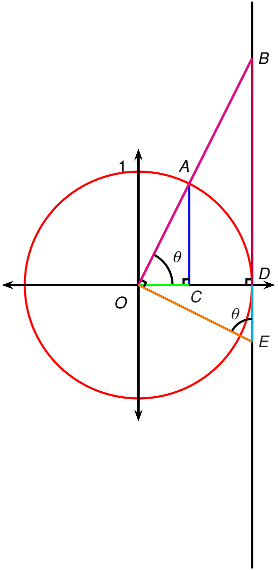
Outline

1. Review of trigonometry
   - The Trigonometric Functions
   - Trigonometric Identities
   - Trigonometric Identities and Complex Numbers
   - Graphs of the Trigonometric Functions
2. Inverse Trigonometric Functions

Geometric interpretation of all trigonometric functions



On picture: circle of radius 1 centered at point O with coordinates $(0, 0)$. Let $\angle DOB = \theta$. Let OB intersect the circle at point A. Then the coordinates of A are $(\cos\theta, \sin\theta)$.

$$\sin\theta = \frac{|AC|}{|OA|} = \frac{|AC|}{1} = |AC|.$$

$$\cos\theta = \frac{|OC|}{|OA|} = \frac{|OC|}{1} = |OC|.$$

$$\tan\theta = \frac{|BD|}{|OD|} = \frac{|BD|}{1} = |BD|.$$

$$\cot\theta = \frac{|DE|}{|OD|} = \frac{|DE|}{1} = |DE|.$$

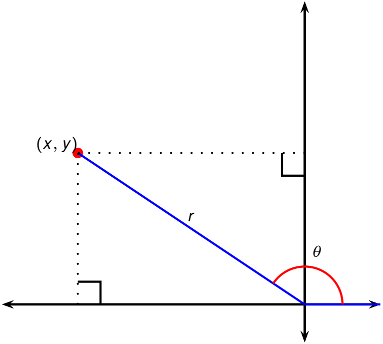
$$\sec\theta = \frac{|OB|}{|OD|} = \frac{|OB|}{1} = |OB|.$$

$$\csc\theta = \frac{|OE|}{|DO|} = \frac{|OE|}{1} = |OE|.$$

Trigonometric Identities

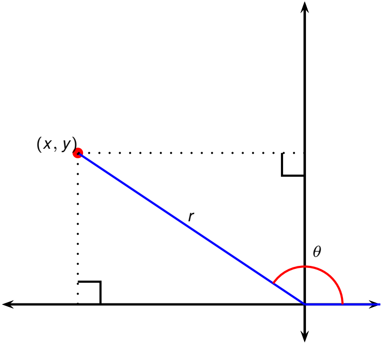
Definition (Trigonometric Identity)

A trigonometric identity is a relationship among the trigonometric functions that is true for any value of the independent variable.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

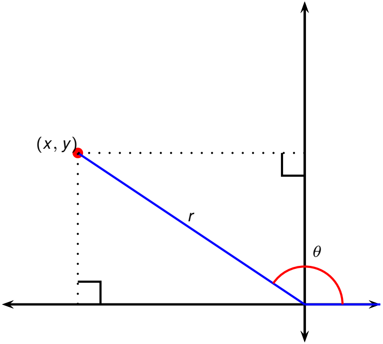
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \ & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \ & \frac{y^2 + x^2}{r^2} \\ = \ & \frac{r^2}{r^2} \\ = \ & 1 \end{aligned}$$

Therefore $\sin^2\theta + \cos^2\theta = 1$.

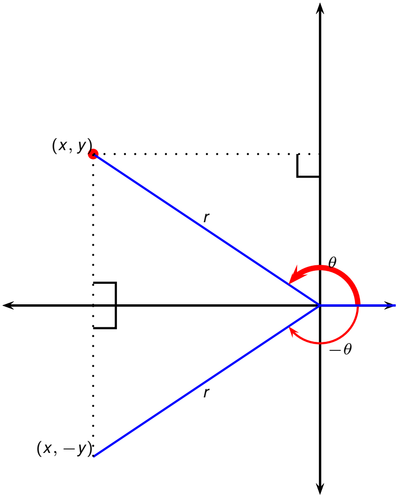


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

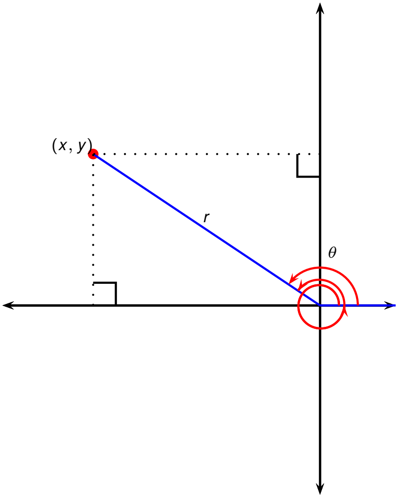
Prove the identity $\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.
- cos is an even function.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say sin and cos are 2π-periodic.

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Substitute $-y$ for y, and use the fact that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$:

$$\begin{aligned} \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \end{aligned}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}$$

To get the half-angle formulas, solve these equations for $\cos^2 x$ and $\sin^2 x$ respectively.

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Divide the first equation by the second, and then cancel $\cos x \cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Do the same for the subtraction formulas:

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Complex numbers definition

Definition

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit.

- Complex numbers are added/subtracted according to the rule
$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$
- Complex numbers are multiplied according to the rule
$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (bc + ad)i \quad .$$

You will not be tested on the material in the following slide.

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Euler's Formula

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Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclllllll}
i \sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & - \dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & + \dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & - \dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. □

You will not be tested on the material in the following slide.

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x, \cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \sin y \cos x \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i\sin(x+y) \\ e^{ix}e^{iy} &= \cos(x+y) + i\sin(x+y) \\ (\cos x + i\sin x)(\cos y + i \sin y) &= \cos(x+y) + i\sin(x+y) \\ \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x) &= \cos(x+y) + i\sin(x+y) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned} 1 &= e^0 \\ &= e^{ix-ix} = e^{ix} e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\ &= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\ &= \cos^2 x + \sin^2 x \quad . \end{aligned}$$

□

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix}e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

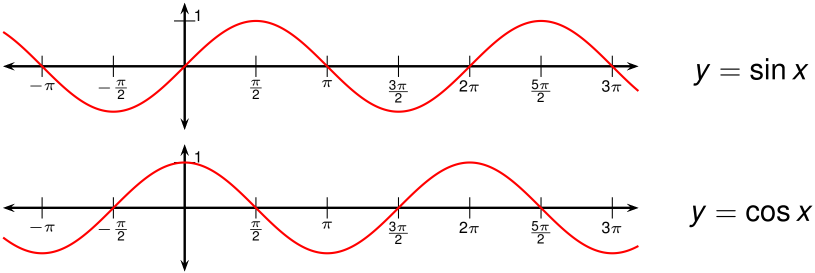
$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \quad . \end{aligned}$$

Proof.

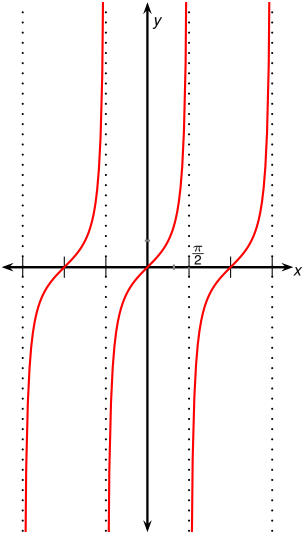
$$\begin{aligned} e^{i(2x)} &= \cos 2x + i \sin 2x \\ e^{ix}e^{ix} &= \cos 2x + i \sin 2x \\ (\cos x + i \sin x)^2 = (\cos x + i \sin x)(\cos x + i \sin x) &= \cos 2x + i \sin 2x \\ \cos^2 x - \sin^2 x + i(2 \sin x \cos x) &= \cos 2x + i \sin 2x \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

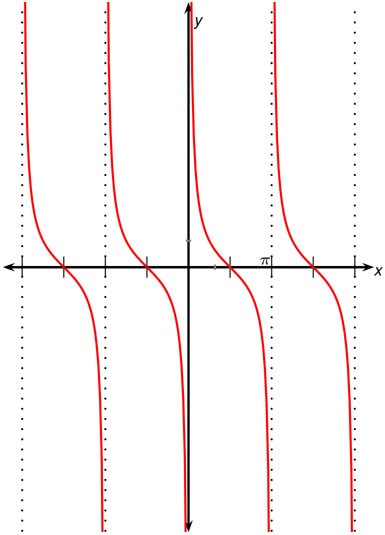
Graphs of the Trigonometric Functions



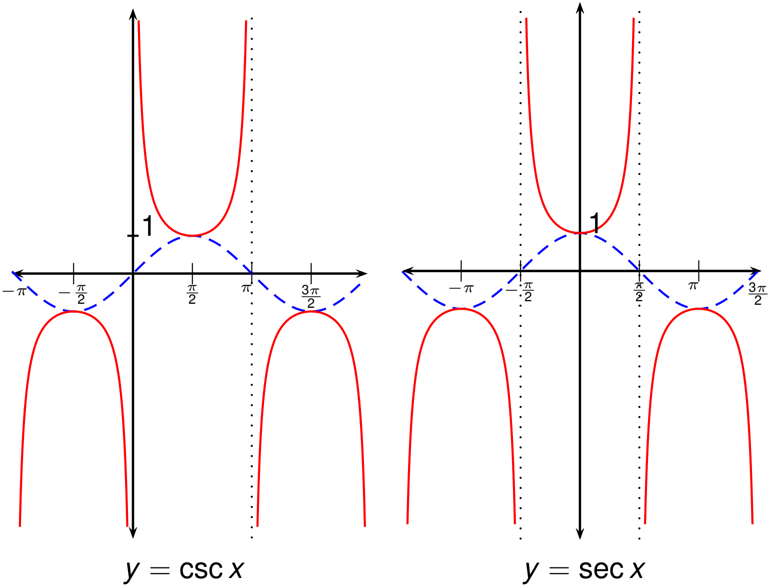
- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\pi/2 + n\pi$ for all integers n.
- $-1 \le \sin x \le 1$.
- $-1 \le \cos x \le 1$.



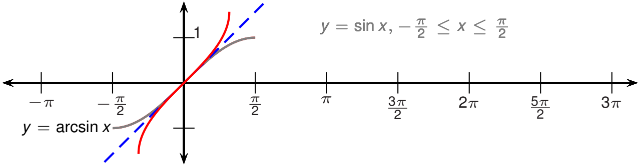
$y = \tan x$



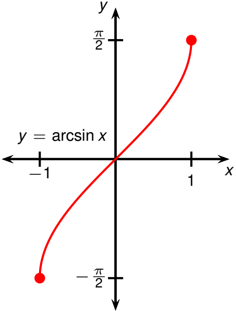
$y = \cot x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
$y = \csc x$
1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
$y = \sec x$

Inverse Trigonometric Functions



- $\sin x$ isn't one-to-one.
- It is if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- Then it has an inverse function.
- We call it arcsin or $\sin^{-1}$.
- $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



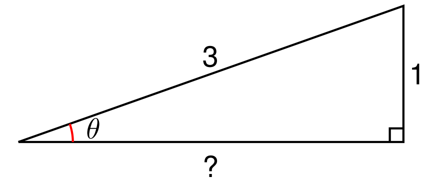
Example

Find $\arcsin\left(\frac{1}{2}\right)$

- $\sin(\quad) = \frac{1}{2}$.
- $-\frac{\pi}{2} \leq \quad \leq \frac{\pi}{2}$.
- Therefore $\arcsin\left(\frac{1}{2}\right) = \quad$.

Find $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$

- Let $\theta = \arcsin(1/3)$, so $\sin\theta = 1/3$.
- Draw a right triangle with opposite side 1 and hypotenuse 3.
- Length of adjacent side $=$
- Then $\tan(\arcsin\frac{1}{3}) =$



Example

Find $\arcsin(\sin 1.5)$.

- $\frac{\pi}{2} \approx 1.57$.
- Therefore $-\frac{\pi}{2} \leq 1.5 \leq \frac{\pi}{2}$.
- Therefore $\arcsin(\sin 1.5) = 1.5$.

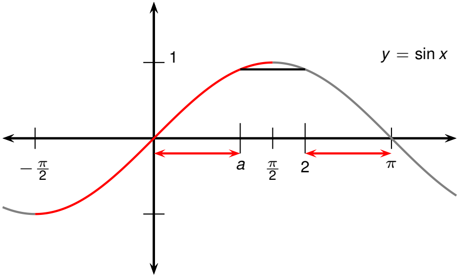
Example

Find $\arcsin(\sin 2)$.

- 2 is not between $-\pi/2$ and $\pi/2$.
- $\sin 2 = \sin a$ for some a between $-\pi/2$ and $\pi/2$.

$$a - 0 = \pi - 2.$$

$$\text{Therefore} \quad \arcsin(\sin 2) = \arcsin(\sin a) = a = \pi - 2.$$



Theorem (The Derivative of arcsin x)

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

Proof.

$$\text{Let} \quad y = \arcsin x.$$

$$\text{Then} \quad \sin y = x \quad \text{and} \quad -\pi/2 \le y \le \pi/2.$$

$$\text{Differentiate implicitly:} \quad \cos y \cdot y' = 1$$

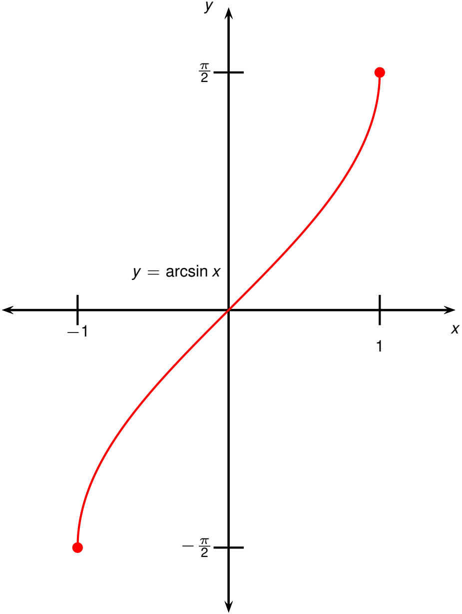
$$y' = \frac{1}{\cos y}$$

$$= \frac{1}{\pm\sqrt{1-\sin^2 y}}$$

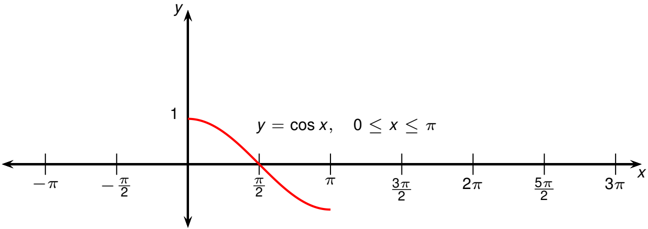
$$\text{But } \cos y > 0: \quad = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

□

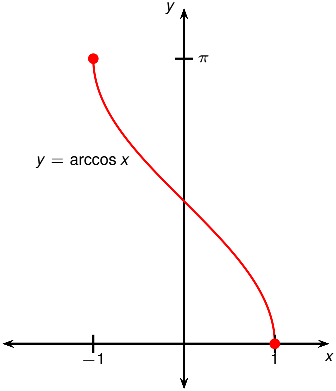
Important facts about arcsin:



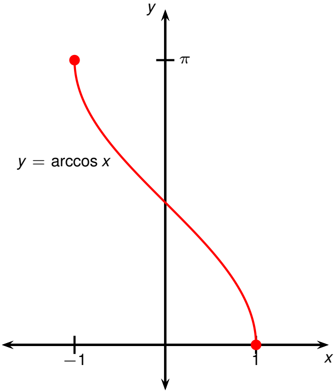
1. Domain:
2. Range:
3. $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
5. $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.



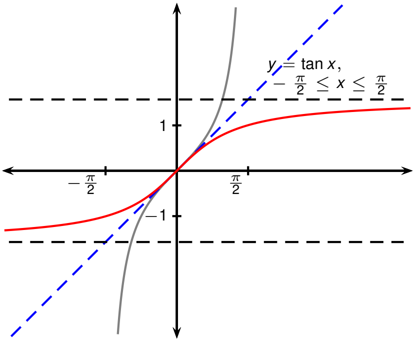
- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.
- $\arccos(x) = y \Leftrightarrow \cos y = x$ and $0 \le y \le \pi$.



Important facts about arccos:



1. Domain:
2. Range:
3. $\arccos x = y \Leftrightarrow \cos y = x$ and $0 \le y \le \pi$.
4. $\arccos(\cos x) = x$ for $0 \le x \le \pi$.
5. $\cos(\arccos x) = x$ for $-1 \le x \le 1$.
6. $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$. (The proof is similar to the proof of the formula for the derivative of $\arcsin x$.)

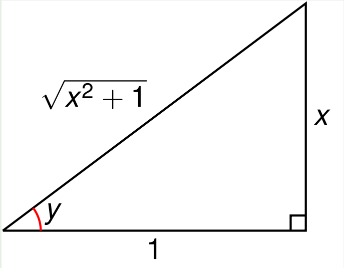


- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The inverse is called $\tan^{-1}$ or arctan.
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of arctan:
- Range of arctan:
- $\lim_{x \to \infty} \arctan x =$
- $\lim_{x \to -\infty} \arctan x =$

Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.
- Then $\cos(\arctan x) =$



Example

Evaluate

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right).$$

$$\frac{1}{x-2} \to \infty \qquad \text{as} \qquad x \to 2^+.$$

Therefore

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) =$$

Theorem (The Derivative of arctan x)

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Proof.

$$\begin{aligned}
\text{Let} \quad y &= \arctan x. \\
\text{Then} \quad \tan y &= x. \\
\text{Differentiate implicitly:} \quad \sec^2 y \cdot y' &= 1 \\
y' &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

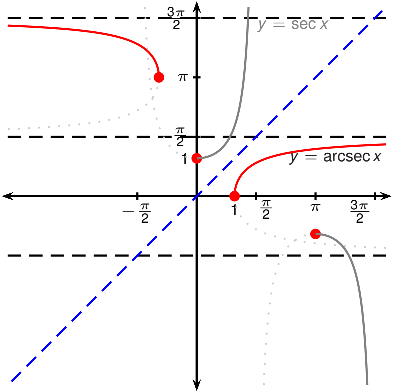
□

The remaining inverse trigonometric functions aren't used as often:

$$
\begin{array}{lllll}
y = \operatorname{arccsc} x \quad (|x| \geq 1) & \Leftrightarrow & \csc y = x & \text{and} & y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\
y = \operatorname{arcsec} x \quad (|x| \geq 1) & \Leftrightarrow & \sec y = x & \text{and} & y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
y = \operatorname{arccot} x \quad (|x| \in \mathbb{R}) & \Leftrightarrow & \cot y = x & \text{and} & y \in (0, \pi)
\end{array}
$$

We will however make use of $\operatorname{arcsec} x$: we discuss in detail its domain.

$$y = \operatorname{arcsec} x \quad (|x| \geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y \in \textbf{?} \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$



- Plot $\sec x$.
- Restrict domain to make one-to-one: Two common choices: $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is good because the domain is easiest to remember: an interval without a point. **NOT our choice.**
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ is good because $\tan x$ is positive on both intervals, resulting in easier differentiation and integration formulas. **Our choice.**

Table of derivatives of inverse trigonometric functions:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

Example (Chain Rule)

$$\text{Differentiate} \quad y = \frac{1}{\arcsin x}.$$

$$\text{Let} \quad u =$$

$$\text{Then} \quad y =$$

$$\text{Chain Rule:} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \left(\qquad \right)\left(\qquad \qquad \right)$$

$$=$$

All of the inverse trigonometric derivatives also give rise to integration formulas. These two are the most important:

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin x + C.$$

$$\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C.$$