

Math 141

Lecture 2

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Spring 2015

Outline

- 1 Integration, Review
 - The Evaluation Theorem (FTC part 2)
- 2 Integration Techniques from Calc I, Review
 - Differential Forms, Review
- 3 Integration and Logarithms, Review

Antiderivatives

Definition (Antiderivative)

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable over $[a, b]$.

In other words, $\int_a^b f(x)dx$ exists for any continuous (over $[a, b]$) function f .

Theorem (The Evaluation Theorem (FTC part 2))

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f .

Indefinite Integrals

- The Evaluation Theorem establishes a connection between antiderivatives and definite integrals.
- It says that $\int_a^b f(x)dx$ equals $F(b) - F(a)$, where F is an antiderivative of f .
- We need convenient notation for writing antiderivatives.
- This is what the indefinite integral is.

Definition (Indefinite Integral)

The indefinite integral of f is another way of saying the antiderivative of f , and is written $\int f(x)dx$. In other words,

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x).$$

Example

$$\int x^4 dx = \frac{x^5}{5} + C$$

because

$$\frac{d}{dx} \left(\frac{x^5}{5} + C \right) = x^4.$$

- The indefinite integral represents a whole family of functions.
- Example: the general antiderivative of $\frac{1}{x}$ is

$$F(x) = \begin{cases} \ln|x| + C_1 & \text{if } x > 0 \\ \ln|x| + C_2 & \text{if } x < 0 \end{cases}$$

- We adopt the convention that the constant participating in an indefinite integral is only valid on one interval.
- $\int \frac{1}{x} dx = \ln|x| + C$, and this is valid either on $(-\infty, 0)$ or $(0, \infty)$.

Differentials

- Recall $\Delta y, \Delta x$ stand for change of x, y . Recall: $\Delta y \approx \frac{dy}{dx} \Delta x$
- $dy = \frac{dy}{dx} dx = dy$
- If we substitute Δy by the formal expression dy and Δx by the formal expression dx , the expression dx appears to “cancel” to give a formal identity.
- Define the *differential* d and the *differential forms* $dx, d(f(x))$ by requesting that d and dx satisfy the transformation law

$$d(f(x)) = f'(x)dx$$

for any differentiable function $f(x)$. In abbreviated notation:

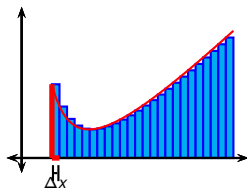
$$df = f'dx$$

Expressions containing expression of the form $d(\text{something})$ are called differential forms.

- $df(x) = f'(x)dx$.
- On the previous slide we stated the differential d and the differential forms dx , $df(x)$ are formal expressions related by a transformation law.
- The precise definitions of differential forms and differentials are outside of the scope of Calculus I and II.
- Differential forms “encode” linear approximations which in turn “encode” “infinitesimal” lengths of segments.
- Courses such as “Integration and Manifolds” or “Differential geometry” usually give precise definitions and fill in the details.
- Nonetheless, what we studied is completely sufficient for practical purposes and carrying out computations.
- **Do not confuse differentials with derivatives.** The correct equality is this.

~~$$df(x) = f'(x)$$~~

$$df(x) = f'(x)dx$$



- $\int_a^b f(x)dx$ is the definite integral of f .
- $\int f(x)dx =$ corresponding anti-derivative.
- \int stands for the limit of a Riemann sum (sum of approximating rectangles).
- dx “encodes” the base length of “infinitesimally small” approximating rectangle, $f(x)$ is the height.
- $f(x)dx$ is a differential form as discussed already.
- We postponed a formal definition of differential form to another course, but we showed how to compute with those.
- This is consistent: integrals of equal differential forms are equal (follows from Net Change Theorem (subst. rule)).

- All rules for computing with derivatives have analogues for computing with differential forms.
- The rules for computing differential forms are a direct consequence of the corresponding derivative rules and the transformation law $d(f(x)) = f'(x)dx$.

Let c be a constant.

Differential rule

$$d(fg) = gdf + f dg$$

$$dc = 0$$

$$d(cf) = c \, df$$

$$d(f + g) = df + dg$$

$$\begin{aligned} df(g(x)) &= f'(g(x))dg(x) \\ &= f'(g(x))g'(x)dx \end{aligned}$$

$$df(g) = f'(g)dg$$

$$dx^n = nx^{n-1}dx$$

$$de^x = e^x dx$$

$$d \sin x = \cos x dx$$

$$d \cos x = -\sin x dx$$

$$d \ln x = \frac{1}{x} dx$$

Derivative rule

$$(fg)' = f'g + fg'$$

$$(c)' = 0$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f(g(x)))' = f'(g(x))g'(x)$$

$$(x^n)' = nx^{n-1}$$

$$(e^x)' = e^x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\ln x)' = \frac{1}{x}$$

Let c be a constant.

Corresponding integration rules. Integration rules justified via the Fundamental Theorem of Calculus

Integration rule

$$\int d(fg) = \int gdf + \int fdg$$

$$\int dc = 0$$

$$\int d(cf) = c \int df$$

$$\int d(f+g) = \int df + \int dg$$

$$\begin{aligned} \int df(g(x)) &= \int f'(g(x))dg(x) \\ &= \int f'(g(x))g'(x)dx \end{aligned}$$

$$\int df(g) = \int f'(g)dg$$

$$dx^n = nx^{n-1}dx$$

$$de^x = e^x dx$$

$$d \sin x = \cos x dx$$

$$d \cos x = -\sin x dx$$

$$d \ln x = \frac{1}{x} dx$$

Derivative rule

$$(fg)' = f'g + fg'$$

$$(c)' = 0$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f(g(x)))' = f'(g(x))g'(x)$$

$$(x^n)' = nx^{n-1}$$

$$(e^x)' = e^x$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\ln x)' = \frac{1}{x}$$

We recall from previous slides that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}.$$

This formula has a special application to integration:

Theorem (The Integral of $1/x$)

$$\int \frac{1}{x} dx = \ln |x| + C.$$

This fills in the gap in the rule for integrating power functions:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Now we know the formula for $n = -1$ too.