Math 141

Lecture 3

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Outline

Integration by Parts

Every differentiation rule corresponds to a differential form rule which in turn corresponds to an integration rule.

$(uv)'$	$=$	$u'v + uv'$	Product Rule
$\mathrm{d}(uv)$	$=$	$v\mathrm{d}u + u\mathrm{d}v$	Differential Prod. Rule
$\int \mathrm{d}(uv)$	$=$	$\int v\mathrm{d}u + \int u\mathrm{d}v$	integration of the above
uv	$=$	$\int v\mathrm{d}u + \int u\mathrm{d}v$	rearrange
$\int u\mathrm{d}v$	$=$	$uv - \int v\mathrm{d}u$	

We just proved the following.

Proposition ((Rule of) Integration by Parts)

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

Example

$$\begin{aligned}\int x\sin x\mathrm{d}x &= \int x\mathrm{d}(-\cos x) \\ &= x(-\cos x) - \int (-\cos x)\mathrm{d}x \\ &= -x\cos x + \int \cos x\mathrm{d}x \\ &= -x\cos x + \sin x + C\end{aligned}$$

$\sin x\mathrm{d}x = \mathrm{d}(-\cos x)$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u.$

Example

$$\begin{aligned} \int \ln x \mathrm{d}x &= (\ln x)x - \int x \mathrm{d}(\ln x) & \text{integrate by parts} \\ &= x \ln x - \int x (\ln x)' \mathrm{d}x \\ &= x \ln x - \int x \frac{1}{x} \mathrm{d}x \\ &= x \ln x - \int \mathrm{d}x \\ &= x \ln x - x + C \quad . \end{aligned}$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$.

Example

$$\begin{aligned}
\int t^2 e^t \mathrm{d}t &= \int t^2 \mathrm{d}\left(e^t\right) && \Big| \text{ integrate by parts} \\
&= t^2 e^t - \int e^t \mathrm{d}\left(t^2\right) \\
&= t^2 e^t - \int e^t 2t \mathrm{d}t \\
&= t^2 e^t - 2\int t \mathrm{d}(e^t) && \Big| \text{ integrate by parts} \\
&= t^2 e^t - 2\left(te^t - \int e^t \mathrm{d}t\right) \\
&= t^2 e^t - 2te^t + 2e^t + C
\end{aligned}$$

Integration by parts: $\int u dv = uv - \int v du.$

Example

$$\begin{aligned}
\int e^x \sin x dx &= \int \sin x \, d\left(e^x\right) \\
&= (\sin x)e^x - \int e^x d(\sin x) \\
&= e^x \sin x - \int e^x \cos x dx \\
&= e^x \sin x - \int \cos x d\left(e^x\right) \\
&= e^x \sin x - \left((\cos x)e^x - \int e^x d(\cos x)\right) \\
&= e^x \sin x - \cos x e^x + \int e^x(-\sin x) dx \\
&= e^x \sin x - \cos x e^x - \int e^x \sin x dx
\end{aligned}$$

Integration by parts: $\int u dv = uv - \int v du.$

Example

$$\begin{aligned}
\int e^x \sin x dx &= e^x \sin x - \cos x e^x - \int e^x \sin x dx \\
2\int e^x \sin x dx &= e^x \sin x - \cos x e^x \\
\int e^x \sin x dx &= \frac{1}{2}\left(e^x \sin x - \cos x e^x\right) + C
\end{aligned}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

Set $w = 1 + x^2$.

$$
\begin{aligned}
\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right) \\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2}\mathrm{d}\left(\frac{x^2}{2}\right) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{1+x^2}\mathrm{d}(1+x^2) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{w}\mathrm{d}w = \frac{\pi}{4} - \frac{1}{2}\left[\ln|w|\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}\left(\ln 2 - \ln 1\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad .
\end{aligned}
$$