Freecalc
Homework on Lecture 11
Quiz date will be announced in class

1. Express the infinite decimal number as a rational number.

   (a) $1.\overline{6} = 1.6666\ldots$

   answer: $\frac{5}{3}$

   (b) $1.\overline{3} = 1.3333\ldots$

   answer: $\frac{4}{3}$

   (c) $2.\overline{16} = 2.16161616\ldots$

   answer: $\frac{214}{99}$

   (d) $2014.\overline{2014} = 2014.2014201420142014\ldots$

   answer: $\frac{20140000}{9999}$

2. **This problem type may appear on the final or the test but will not appear on the quiz.** Express the sum of the series as a rational number.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$

   answer: $\frac{13}{6}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5^n}{10^n}$

   answer: $\frac{13}{4}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n-3^n}{7^n}$

   answer: $\frac{7}{4}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}+7^{n-1}}{21^n}$

   answer: $\frac{4}{7}$

Solution. 2.a

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n} &= \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n \\
&= \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n + \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n && \left|\ \text{Use geometric series sum f-la: } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \text{ provided } |r|<1\right. \\
&= \frac{2}{5}\frac{1}{\left(1-\frac{2}{5}\right)} + \frac{3}{5}\frac{1}{\left(1-\frac{3}{5}\right)} \\
&= \frac{13}{6} \quad .
\end{aligned}$$

Solution. 2.b

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n+5^n}{10^n} &= \sum_{n=0}^{\infty}\left(\frac{1}{5^n}+\frac{1}{2^n}\right) && \left|\ \text{use } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \text{ for } |r|<1\right. \\
&= \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{5}} \\
&= \frac{13}{4} \quad .
\end{aligned}$$

Solution. 2.d

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n+1}+7^{n-1}}{21^n} &= \sum_{n=1}^{\infty}\left(3\frac{3^n}{21^n}+\frac{1}{7}\frac{7^n}{21^n}\right) \\
&= 3\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{7}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n \\
&= \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{21}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n && \Big|\ \text{use } \textstyle\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1 \\
&= \frac{3}{7}\frac{1}{\left(1-\frac{1}{7}\right)}+\frac{1}{21}\frac{1}{\left(1-\frac{1}{3}\right)} \\
&= \frac{4}{7} \quad .
\end{aligned}$$

3. Sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{-6}{9n^2+3n-2}$.

answer: 2

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2}$.

answer: 3

(c) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

answer: $-\ln 2$

Solution. 3.b

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2} &= \sum_{n=3}^{\infty}\left(\frac{3}{n-2}-\frac{3}{n-1}\right) && \Big|\ \text{use partial fractions, see below} \\
&= 3\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right) \\
&= 3\left(\underbrace{\left(1-\frac{1}{2}\right)}_{n=3}+\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=4}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=5}+\dots\right) \\
&= 3\lim_{n\to\infty}\left(1-\frac{1}{n-1}\right)=3 \quad .
\end{aligned}$$

In the above we used the partial fraction decomposition of $\frac{3}{n^2-3n+2}$. This decomposition is computed as follows.

$$\frac{3}{n^2-3n+2}=\frac{3}{(n-1)(n-2)}$$

We need to find A_i's so that we have the following equality of rational functions. After clearing denominators, we get the following equality.

$$3=A_1(n-2)+A_2(n-1)$$

After rearranging we get that the following polynomial must vanish. Here, by "vanish" we mean that the coefficients of the powers of x must be equal to zero.

$$(A_2+A_1)n+(-A_2-2A_1-3)$$

In other words, we need to solve the following system.

$$\begin{array}{lll} -2A_1 & -A_2 & =3 \\ A_1 & +A_2 & =0 \end{array}$$

System status	Action

$-2A_1 \quad -A_2 \quad = 3$ $A_1 \quad +A_2 \quad = 0$	Selected pivot column 2. Eliminated the non-zero entries in the pivot column.
$A_1 \quad +\frac{A_2}{2} \quad = -\frac{3}{2}$ $\frac{A_2}{2} \quad = \frac{3}{2}$	Selected pivot column 3. Eliminated the non-zero entries in the pivot column.
$A_1 \quad = -3$ $A_2 \quad = 3$	Final result.

Therefore, the final partial fraction decomposition is the following.

$$\frac{3}{n^2 - 3n + 2} = \frac{-3}{(n-1)} + \frac{3}{(n-2)}.$$

Solution. 3.c.

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) &= \sum_{n=2}^{\infty} \left(\ln\left(1 - \frac{1}{n}\right) + \ln\left(1 + \frac{1}{n}\right)\right) \\
&= \sum_{n=2}^{\infty} \left(\ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)\right) \\
&= \sum_{n=2}^{\infty} \left(\ln(n-1) - 2\ln(n) + \ln(n+1)\right) \\
&= (\ln 1 - 2\ln 2 + \cancel{\ln 3}) + (\ln 2 \cancel{- 2\ln 3} + \cancel{\ln 4}) \\
&\quad + (\cancel{\ln 3} \cancel{- 2\ln 4} + \cancel{\ln 5}) + \cancel{\dots}\dots \\
&= \lim_{n\to\infty} \left(-\ln 2 - \ln n + \ln(n+1)\right) \\
&= \lim_{n\to\infty} \left(-\ln 2 + \ln\left(\frac{n+1}{n}\right)\right) \\
&= -\ln 2 \quad .
\end{aligned}$$

4. **Problems b, c, d will not appear on the quiz.** Use partial fractions to sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

answer: 1

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{2n+1}{n^4 + 2n^3 - n^2 - 2n}$

answer: $\frac{1}{3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^4 - 3n^2 + 1}$

answer: -1

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2 + n + 2}{n^4 - 5n^2 + 4}$

answer: $\frac{3}{4}$

Solution. 4d

The partial fractions decomposition algorithm shows that

$$\frac{n^2 + n + 2}{n^4 - 5n^2 + 4} = \frac{1}{3}\left(\frac{2}{n-2} - \frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{n+2}\right) \quad .$$

We omit the details of the partial fraction decomposition as it is quite laborious, but otherwise straightforward.

Therefore

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{n^2+n+2}{n^4-5n^2+4} &= \frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{2}{n-2}-\frac{2}{n-1}+\frac{1}{n+1}-\frac{1}{n+2}\right)\\
&= \frac{2}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)\\
&\quad +\frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\\
&= \frac{2}{3}\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{2}{3}\right)+\ldots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\ldots\right)\\
&\quad +\frac{1}{3}\left(\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\ldots\right)\\
&= \lim_{n\to\infty}\frac{2}{3}\left(1-\frac{1}{n-1}\right)+\lim_{n\to\infty}\frac{1}{3}\left(\frac{1}{4}-\frac{1}{n+2}\right)\\
&= \frac{2}{3}+\frac{1}{3}\cdot\frac{1}{4}\\
&= \frac{3}{4}\quad .
\end{aligned}$$