# Math 141

## Lecture 11

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# Outline

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# Formal Series

## Definition (Formal Series)

A formal series is a list of numbers delimited by the plus sign.

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

- Recall a sequence is a list of numbers.

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \ldots, \quad a_n, \quad \ldots$$

- The $+$ sign indicates our intention to attempt to sum the elements of the formal series.
- Except for the indication of that intention, formal series and sequences are essentially synonymous.
- The sum of a finite sequence/finite formal series is studied in the subject of elementary arithmetics.
- The sum, if convergent, of an infinite sequence/infinite formal series will be defined in the following slides.

## Example (The $\ldots$ and $\sum$ notations for series)

Let $A$ be the sum of the positive even integers between 2 and and 124. Write $A$ using the $\ldots$ notation and using the $\sum$ notation.

$$\begin{aligned} A &= 2+4+6+\cdots+124 \\ &= 2+4+6+\cdots+2n+\cdots+124 \\ &= 2\cdot 1+2\cdot 2+2\cdot 3+\cdots+2\cdot n+\cdots+2\cdot 62 \\ &= \sum_{n=1}^{62} 2n \quad . \end{aligned}$$

- We aim to introduce the $\sum$ notation for series via this example.
- The $\ldots$ notation is informal but easier to read.
- If the $\ldots$ are too ambiguous, we should include the general term.
- To make it clearer we should rewrite all elements in the pattern of the general term.
- If that is still ambiguous we should switch to the completely unambiguous $\sum$ notation.

## Example (The $\ldots$ and $\sum$ notations for series)

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- The number $n$ is the index (counter) of the sum.
- $\sum$ tells us to add several copies of the summed term, where in each term the index is replaced by a concrete value.
- The values taken by the index are determined by the boundaries of summation.
- The index varies over all integers starting with the lower boundary and ending with upper boundary.
- In programming, where do we use notation similar to that for $\sum$?

## Example (The $\ldots$ and $\sum$ notations for series)

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- To go from $\sum$ to $\ldots$ notation: substitute few values for the index. Make sure to include the last value.
- To go from $\ldots$ to $\sum$ notation:
  - figure out a pattern for the general term just as with sequences;
  - select first and last index so that your general term formula reproduces the first and last terms of the sequence.

## Example (The $\ldots$ and $\sum$ notations for series)

Let $A$ be the sum of the positive even integers between 2 and and 124. Write $A$ using the $\ldots$ notation and using the $\sum$ notation.

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- Bear in mind the $\ldots$ notation is informal.
  - There are infinitely many formulas that fit any single pattern.
  - Thus it is acceptable to use the $\ldots$ notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt or seeking complete rigor we should use the $\sum$ notation.

## Definition (Arithmetic series)

An arithmetic series is a series whose terms are an arithmetic sequence.

## Example (Sum of a small arithmetic series)

The sum of the arithmetic series $7+4+1-2-5$ is

## Example (Sum of a large arithmetic series)

Find the sum of the arithmetic series

$$7+4+1-2-5-\cdots-53-56.$$

Let $s$ denote the sum.

$$\begin{array}{rcrrrrr} s & = & 7 & +4 & +1 & -\cdots & -56 \\ +s & = & -56 & -53 & -50 & -\cdots & +7 \\ \hline 2s & = & -49 & -49 & -49 & -\cdots & -49 \end{array}$$

$$\begin{aligned} \text{Therefore}\quad 2s &= (-49)(22) \\ s &= -49\cdot 22/2 = -539. \end{aligned}$$

## Theorem (Sum of an arithmetic series)

*The sum of a finite arithmetic series is the average of the first and last terms, multiplied by the number of terms. That is,*

$$a + (a + d) + (a + 2d) + \cdots + (a + (n-1)d) = \frac{a + (a + (n-1)d)}{2} n.$$

*The only infinite arithmetic series with a sum is the series of all* $0$.

## Example (Sum of an arithmetic series)

Find the sum of the arithmetic series

$$5 + 10 + 15 + 20 + \cdots + 100.$$

The series contains terms. The average of the first and last terms is .

Therefore the sum is $\frac{}{2}$ .

## Definition (Geometric series)

A geometric series is a series whose terms are a geometric sequence.

## Example (The sum of a finite geometric series)

Let $r \neq 1$. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{M-1} = \sum_{n=1}^{M} ar^{n-1}.$$

Let $s$ denote the sum.

$$\begin{array}{rcllllll} s &=& a & +ar & +ar^2 & +\cdots & +ar^{M-1} & \\ - \quad rs &=& & ar & +ar^2 & +\cdots & +ar^{n-1} & +ar^M \\ \hline s - rs &=& a - ar^M & & & & & \\ s &=& \frac{a(1-r^M)}{1-r} & & & & & \end{array}$$

## Theorem (The sum of a finite geometric series)

*Let $r \neq 1$. The sum of the finite geometric series $\sum_{n=1}^{M} ar^{n-1}$ is $a\frac{1-r^M}{1-r}$.*

- Does it make sense to add infinitely many numbers?
- Sometimes yes, sometimes no.
- Consider the series $\sum_{n=1}^{\infty} n$.

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

- If we add the terms, we get the partial sums 1,3,6,10,15.
- After the $n$th term, we get $\frac{n(n+1)}{2}$.
- This goes to $\infty$ as $n$ gets bigger.
- Now consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$$

- If we add the terms, we get the partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$.
- After the $n$th term, we get $1 - \frac{1}{2^n}$.
- This gets closer and closer to 1. We write $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

## Definition (Partial Sum, Convergent, Divergent, Sum)

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote the $n$th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s.$$

In this case, we call $s$ the sum of the series.
If the sequence $\{s_n\}$ is divergent, then we say that the series $\sum_{i=1}^{\infty} a_i$ is divergent.

## Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

- If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$.
- Since $\lim_{n\to\infty} s_n$ doesn't exist, the series is divergent when $r = 1$.
- If $r \neq 1$, then

$$\begin{array}{rrcllllll} & s_n & = & a & +ar & +ar^2 & +\cdots & +ar^{n-1} & \\ - & rs_n & = & & ar & +ar^2 & +\cdots & +ar^{n-1} & +ar^n \\ \hline & s_n - rs_n & = & a - ar^n & & & & & \\ & s_n & = & \frac{a(1-r^n)}{1-r} & & & & & \end{array}$$

- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then $r^n$ is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

## Theorem (Convergence of Geometric Series)

*The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

*is convergent if* $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

*If* $|r| \geq 1$*, the series is divergent.*
$a$ *is called the first term and* $r$ *is called the common ratio.*

For $r \neq 1$, recall that the sum of a geometric series is

$$a + ar + ar^2 + ar^3 + \dots \quad = \quad a\left(1 + r + r^2 + r^3 + \dots\right) \quad = \quad \frac{a}{1-r}$$

alternatively

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{m=0}^{\infty} ar^m \quad = \quad a\sum_{m=0}^{\infty} r^m \quad = \quad \frac{a}{1-r}$$

## Example

Find the sum of the geometric series $\quad -2 + \frac{6}{5} - \frac{18}{25} + \frac{54}{125} - \cdots$

- The first term is $a = -2$.
- The common ratio is $r = \frac{\frac{6}{5}}{-2} = -\frac{3}{5}$.
- Therefore the sum is

$$\sum_{n=1}^{\infty} (-2)\left(-\frac{3}{5}\right)^{n-1} = \frac{(-2)}{1-\left(-\frac{3}{5}\right)} = -\frac{2}{\frac{8}{5}} = -\frac{5}{4}$$

## Example

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a quotient of integers.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

- After the first term, we have a geometric series.
- $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

$$\begin{aligned} 2.3171717\ldots &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

## Example

Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Is this a geometric series? No, because $\displaystyle\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \frac{n}{n+2}$

is not constant. Decompose $a_n$ into partial fractions:

$$
\begin{aligned}
a_n &= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\
s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{k+1}
\end{aligned}
$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$

## Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2} \\
&\vdots \\
s_{2^n} &> 1 + \tfrac{n}{2}
\end{aligned}$$

Therefore $s_{2^n} \to \infty$ as $n \to \infty$, so $\{s_n\}$ is divergent, so the harmonic series is divergent.