Math 141
Lecture 14

Greg Maloney

Todor Milev

University of Massachusetts Boston

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Outline

1. Power Series

2. Power Series as Functions
   - Differentiation and Integration of Power Series

3. Taylor and Maclaurin Series

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Power Series

Definition (Power Series)

A power series is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.
- A power series might converge for some values of x and diverge for others.
- The sum of the series is a function
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
whose domain is the set of all x for which the series converges.
- f resembles a polynomial, except it has infinitely many terms.

Definition (Power Series Centered at *a*)

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at *a* or a power series about *a* or a power series in $(x-a)$.

- We use the convention that $(x-a)^0 = 1$, even if $x = a$.
- If $x = a$, then all terms are 0 for $n \geq 1$, so the series always converges when $x = a$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n! x^n$.
- If $x \neq 0$, then

$$
\begin{aligned}
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\
&= \lim_{n \to \infty} (n+1)|x| \\
&= \infty
\end{aligned}
$$

- Therefore by the Ratio Test the series diverges for all $x \neq 0$.
- Therefore the series only converges for $x = 0$.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.

- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$
$$= \lim_{n \to \infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.

- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is convergent.

- The series converges if $2 \leq x < 4$ and diverges otherwise.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{x^2}{4(n+1)^2} = 0 < 1$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore the domain of the function is $(-\infty, \infty)$, or $\mathbb{R}$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x - a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.*

Definition (Radius of Convergence)

The number R in case three of the theorem is called the radius of convergence of the power series.

1. In the first case, we say $R = 0$.
2. In the second case, we say $R = \infty$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x - a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.*

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

1. In the first case, the interval contains the single point a.
2. In the second case, the interval is $(-\infty, \infty)$.
3. In the third case, the inequality $|x - a| < R$ can be rewritten $a - R < x < a + R$.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus, in the third case, there are four possibilities for the interval of convergence.
  1. $[a - R, a + R)$
  2. $(a - R, a + R]$
  3. $[a - R, a + R]$
  4. $(a - R, a + R)$
- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.
- The Ratio and Root Tests will always fail when x is an endpoint $a - R$ or $a + R$, so the endpoints must be checked with another test.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+1}{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x| \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|
\end{aligned}
$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent.
- The interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Representations of Functions as Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If it converges, the sum is $\frac{1}{1-x}$.
- The thing that is new in this section is the we now regard the series $\sum_{n=0}^{\infty} x^n$ as expressing the function $f(x) = \frac{1}{1-x}$.
- This only works if $-1 < x < 1$.

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)}$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

- Another way to write the series is $\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.
- This converges if $|-x^2| < 1$, that is, if $x^2 < 1$, or $|x| < 1$.
- Therefore the interval of convergence is $(-1, 1)$.

Example

Find a power series representation for $\frac{1}{x+2}$.

$$
\begin{aligned}
\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{\left[1-\left(-\frac{x}{2}\right)\right]} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\
&= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots
\end{aligned}
$$

Interval of convergence:

$$
\begin{aligned}
\left|-\frac{x}{2}\right| &< 1 \\
|x| &< 2
\end{aligned}
$$

Therefore the interval of convergence is $(-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$
\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
\end{aligned}
$$

- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.
- The interval of convergence is again $(-2, 2)$.

Differentiation and Integration of Power Series

Theorem (Differentiation and Integration of Power Series)

If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

1. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$

2. $\displaystyle\int f(x)\, \mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$
   $= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$

- This is called term-by-term differentiation and integration.
- Another way of saying it is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right]$$

$$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]\mathrm{d}x = \sum_{n=0}^{\infty}\int\left[c_n(x-a)^n\right]\mathrm{d}x$$

- We can treat power series like polynomials with infinitely many terms.

Example

Find the derivative of the Bessel function

$$
\begin{aligned}
J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\
J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathsf{d}}{\mathsf{d}x}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}
\end{aligned}
$$

- $J_0(x)$ is defined everywhere.
- Therefore its derivative $J_0'(x)$ is also defined everywhere.

Example

Find a power series for $\ln(1 - x)$ and state its radius of convergence.
$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(1 - x) = -\frac{1}{1 - x}$$

Therefore
$$
\begin{aligned}
\ln(1 - x) &= -\int \frac{1}{1 - x}\mathrm{d}x = -\int (1 + x + x^2 + x^3 + \cdots)\mathrm{d}x \\
&= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \right) + C = C - \sum_{n=1}^{\infty} \frac{1}{n}x^n
\end{aligned}
$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

Therefore
$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{1}{n}x^n$$

Example

Find a power series for arctan x and state its radius of convergence.

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

Therefore

$$\begin{aligned}
\arctan x &= \int \frac{1}{1 + x^2} dx = \int (1 - x^2 + x^4 - x^6 + \cdots) dx \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}
\end{aligned}$$

- The radius is the same as for the original series: $R = 1$.
- To find C, plug in $x = 0$: $0 = C$.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}$$

(12.10) Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + 4 \cdot 5c_5(x - a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots$
- $f'''(a) = 2 \cdot 3c_3 = 3!c_3$.
- $f^{(n)}(a) = n!c_n$.
- Therefore $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Here is what we get if we plug these coefficients into the power series:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots
\end{aligned}$$

Definition (Taylor Series)

This series is called the Taylor series of *f*.

The case when $a = 0$ is special enough to have its own name:

Definition (Maclaurin Series)

The Maclaurin series of f is the Taylor series of f centered at $a = 0$. In other words, it is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.

Example

Find the sum of the series
$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n n!} = 1 - \frac{1}{2 \cdot 1!} + \frac{1}{4 \cdot 2!} - \frac{1}{8 \cdot 3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n n!} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{2}\right)^n \\
&= e^{-1/2} \\
&= \frac{1}{\sqrt{e}}
\end{aligned}
$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.

- $f^{(n)}(3) = e^3$.

- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n} \right| = \lim_{n \to \infty} \frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.

- Therefore $R = \infty$.

- Just like the Maclaurin series, this series also represents e^x.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$
\begin{aligned}
e^x &= e^{x-3+3} = e^3 e^{x-3} \\
&= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n
\end{aligned}
$$

Recall that $e^y = \sum\limits_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

The radius of convergence was already computed to be $R = \infty$.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$
\begin{array}{rclcrcl}
f(x) &=& \sin x & \quad & f(0) &=& 0 \\
f'(x) &=& \cos x & \quad & f'(0) &=& 1 \\
f''(x) &=& -\sin x & \quad & f''(0) &=& 0 \\
f'''(x) &=& -\cos x & \quad & f'''(0) &=& -1 \\
f^{(4)}(x) &=& \sin x & \quad & f^{(4)}(0) &=& 0
\end{array}
$$

The Maclaurin series is

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
$$

Use the Ratio Test to find R.

$$
\begin{aligned}
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\
&= \lim_{n \to \infty} \frac{x^2}{(2n+2)(2n+3)} = 0
\end{aligned}
$$

Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the sum of the series
$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\
&= \sin\frac{\pi}{2}\\
&= 1
\end{aligned}$$

Example (Example 5, p. 776)

Find the Maclaurin series for $\cos x$.

$$
\begin{aligned}
\cos x = \frac{\mathsf{d}}{\mathsf{d}x} \left(\sin x \right) &= \frac{\mathsf{d}}{\mathsf{d}x} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \\
&= \sum_{n=0}^{\infty} \frac{\mathsf{d}}{\mathsf{d}x} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}
$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.

Example (Example 6, p. 776)

Find the Maclaurin series for $x \cos x$.

$$
\begin{aligned}
x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\
&= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots
\end{aligned}
$$

Here is a table of some important Maclaurin series we have learned:

Function	Series	R
$\dfrac{1}{1-x}$	$= \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	1
$\arctan x$	$= \displaystyle\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	1
e^x	$= \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	∞
$\sin x$	$= \displaystyle\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	∞
$\cos x$	$= \displaystyle\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	∞

Example (Example 11, p. 780)

Use a power series to find $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \frac{1}{2}$$

Example

Use a power series to find $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots$$

$$x - \sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots$$

$$\frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots$$

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) = \frac{1}{6}$$